Locally homogeneous structures on Hopf surfaces

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Abstract

We study holomorphic locally homogeneous geometric structures modelled on line bundles over the projective line. We classify these structures on primary Hopf surfaces. We write out the developing map and holonomy morphism of each of these structures explicitly on each primary Hopf surface.

Contents

1 Introduction

1.1 The problem

Definition 1.1. Suppose that M is a manifold and that G/H is a homogeneous space. A *G/H-structure* on M is a maximal choice of coordinates on M valued in G/H, with transition maps given by action of elements of G. A G/H-structure is also called a locally homogeneous structure modelled on G/H.

There is a great deal known about G/H-structures on compact complex surfaces, as long as H is compact; see Wall [31, 32]. We will suppose instead that G/H is a complex homogeneous space (i.e. that G is a complex Lie group and $H \subset G$ is a closed complex Lie subgroup), and that the G/H-structure is holomorphic (i.e. the coordinates valued in G/H are all holomorphic maps). A homogeneous space G/H is *primitive* if G does not preserve a foliation on G/H. A locally homogeneous structure is called *primitive* if its model is. The primitive holomorphic locally homogeneous structures on compact complex surfaces are classified; see Klingler [16]. The imprimitive are a mystery, although the foliations are roughly classified; see Brunella [3]. This paper will classify explicitly a particular family of imprimitive holomorphic locally homogeneous structures (the $\mathcal{O}(n)$-structures) on a particular family of compact complex surfaces (the primary Hopf surfaces). The technique consists largely of elementary power series calculations using Weierstrass polynomials in different coordinate charts. Along the way we develop a systematic machinery for computations on Hopf surfaces. Our aim in this paper is to develop the tools needed to eventually classify all holomorphic locally homogeneous structures on all compact complex surfaces. Although the arguments of this paper are disappointingly

complicated, the results are simple and surprising. The authors believe that uncovering these results is an essential step in the large and important programme of understanding geometry of locally homogeneous structures on low dimensional manifolds.

1.2 Organization of this paper

Before we can study geometric structures on Hopf surfaces, we will need to review the known results on cohomology of line bundles on Hopf surfaces, and also classify the flat $\mathbb{P}^1$-bundles on Hopf surfaces. We complete this in sections 2 and 3.

In section 4, we define the concept of locally homogeneous geometric structure, and we explain the simplifications to the general theory that occur on Hopf surfaces.

The geometric structures in this paper are modelled on the total space $\mathcal{O}(n)$ of the usual holomorphic line bundle $\mathcal{O}(n) \to \mathbb{P}^1$. We write $\mathcal{O}(n)$ as G/H for suitable groups G and H. We explain how $\mathcal{O}(n)$-structures can be encoded as ordinary differential equations in section 4. The group G acting on $\mathcal{O}(n)$ is complicated, and we need to unravel its conjugacy classes in some detail in section 5.

In section 6, we write out explicit expressions in coordinates for each of the $\mathcal{O}(n)$-structures on each Hopf surface. Unfortunately for our study, certain Hopf surfaces (known as hyperresonant Hopf surfaces) have large and complicated families of $\mathcal{O}(n)$-structures, depending on arbitrarily large families of parameters, which are responsible for the length and complexity of this paper. The hyperresonant structures are the only surprise in this paper, having no apparent geometric description. Section 7 proves that the various $\mathcal{O}(n)$-structures that we have explicitly written out are the only ones that any Hopf surface can bear. Section 8 presents some preliminary results on locally homogeneous geometric structures inducing these $\mathcal{O}(n)$-structures.

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2 Definition and survey of Hopf surfaces

2.1 The Poincaré domain

Suppose that $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a biholomorphism fixing the origin. Suppose moreover that all eigenvalues λ of $F'(0)$ satisfy $|\lambda| < 1$; F is said to lie in the *Poincaré domain.* By the Poincaré–Dulac theorem (see [1] p. 192), F is conjugate by a biholomorphism of $\mathbb{C}^2$ to a map of precisely one of the two forms

$$\begin{aligned} F(z) &= (\lambda_1 z_1, \lambda_2 z_2), && 0 < |\lambda_2| \le |\lambda_1| < 1 \text{ or} \\ F(z) &= (\lambda z_1, \lambda^m z_2 + z_1^m), && 0 < |\lambda| < 1, m \ge 1. \end{aligned}$$

In the first case, F is called *diagonal*. In the second case, F is called *exceptional* and m is an integer which we will call the *degree* of F. We will not quite follow [24] in describing a diagonal map as:

homothetic	$\lambda_1 = \lambda_2$	
hyperresonant	$\lambda_1^{m_1} = \lambda_2^{m_2}$	, some $m_1 \geq m_2 \geq 1$ integers
generic	otherwise.	

In this article (breaking from tradition) we will consider homotheties to be special cases of hyperresonant maps, rather than requiring that a hyperresonant map have $m_1, m_2 \geq 2$. If a map F is hyperresonant, then we will refer to the pair m_1, m_2 of integers for which $\lambda_1^{m_1} = \lambda_2^{m_2}$ and for which m_1 (and hence m_2) has the smallest possible positive value, as the *hyperresonance* of F.

2.2 Hopf surfaces

A *Hopf surface* is a compact complex surface covered by $\mathbb{C}^2 \setminus 0$. For example, for any F in the Poincaré domain, let S_F be the quotient $\left(\mathbb{C}^2 \setminus 0\right)/(z \sim F(z))$. Hopf surfaces of the form S_F are called *primary*. Every Hopf surface admits a finite covering by a primary Hopf surface (see [18] p. 696). From now on, when we refer to a Hopf surface, we will always assume that it is primary. Two Hopf surfaces are biholomorphic just when the associated biholomorphisms of $\mathbb{C}^2$ are conjugate by a biholomorphism. Any term used to describe the map F will also be used to describe S_F; for example a Hopf surface is called *linear* or *diagonal* or *resonant*, etc. if the map F is.

An example: if $F(z) = \frac{1}{2}z$, then clearly S_F is diffeomorphic to $S^3 \times S^1$. Every Hopf surface can be smoothly deformed into this one, through a family of Hopf surfaces, so all Hopf surfaces are diffeomorphic to $S^3 \times S^1$.

2.3 Biholomorphism groups of Hopf surfaces

The biholomorphism groups of Hopf surfaces are well known [25, 33]:

$F(z_1, z_2)$	$\operatorname{Bihol} S_F$
homothety	invertible linear maps
nonhomothetic diagonal linear	invertible diagonal linear maps
$(\lambda z_1, \lambda^m z_2 + z_1^m)$	$(z_1, z_2) \mapsto (a z_1, a^m z_2 + b z_1^m)$

with $a \neq 0$ and b arbitrary complex constants.

2.4 Meromorphic functions

Definition 2.1. A Weierstrass polynomial $W(z_1, z_2)$ is a polynomial in z_1, with coefficients holomorphic functions of z_2, so that there is some point of the complex line $z_1 = 0$ at which $W(z_1, z_2) \neq 0$.

Lemma 2.2. *Suppose that $U \subset \mathbb{C}^2$ is an open neighborhood of the origin, and that f is a meromorphic function on $U \setminus 0$. Then there is an open neighborhood $U' \subset \mathbb{C}^2$ of the origin, with $U' \subset U$, an integer k, relatively prime Weierstrass polynomials W_1 and W_2 on U', and a function h holomorphic on U', so that*

$$f = h\, z_1^k \, \frac{W_1}{W_2}$$

in $U' \setminus 0$. The neighborhood U' is not uniquely determined, but once U' is chosen then the rest is uniquely determined, i.e. any two such representations must agree on U'.

Proof. By Levi's theorem (see [29]) any meromorphic function on $U \setminus 0$ extends uniquely to a meromorphic function U. We can then find a possibly smaller neighborhood U' of 0 on which $f = h_1/h_2$ is a ratio of holomorphic functions. Write out Weierstrass polynomial factorizations of h_1 and h_2; these exist by the Weierstrass preparation theorem [12] p. 157. The functions involved are all uniquely determined by uniqueness of Weierstrass polynomial factorization of holomorphic functions. ☐

The meromorphic functions on Hopf surfaces are well known [24]:

Lemma 2.3.

F	$\mathbb{C}\left(S_F\right)$	
hyperresonant	$\mathbb{C}\left(\frac{z_1^{m_1}}{z_2^{m_2}}\right)$	*if* $\lambda_1^{m_1} = \lambda_2^{m_2}$
generic	$\mathbb{C}$	
exceptional	$\mathbb{C}$	

Remark 2.4. An simpler but incorrect proof of this lemma has been given in the literature; [18] p. 697 and [2] p. 226. For F exceptional or generic, the arguments work perfectly well, and yield the indicated meromorphic sections. For F hyperresonant, it turns out that we will need a little more work, as will be clarified below. These authors each claim that if f is a meromorphic function on a Hopf surface, then $z_1^N f$ is holomorphic on $\mathbb{C}^2$, for large enough N, which is not true for

$$f = \frac{z_1 + z_2}{z_1 - z_2}$$

even though this function f is meromorphic on the Hopf surface $S_{1/2}$.

Proof. Suppose that f is a meromorphic function on a Hopf surface S_F. Treat f as an F-invariant meromorphic function on $\mathbb{C}^2$. By lemma 2.2, near the origin $f = \frac{h\, W_1}{W_2}$, with W_1 and W_2 uniquely determined Weierstrass polynomials in z_1, and h nowhere vanishing and holomorphic. Under action of F, these Weierstrass polynomials get transformed into new Weierstrass polynomials in z_1, up to scaling. By uniqueness of Weierstrass polynomials for the numerator and denominator of f, F must just scale each Weierstrass polynomial. But then h must also only get rescaled. Hence h, W_1 and W_2 are themselves sections of various line bundles on the Hopf surface. The value of $h\left(z_1, z_2\right)$ at the origin is

the same nonzero value as that of $h\left(\lambda_1 z_1, \lambda_2 z_2\right)$ at the origin. So h must scale by 1, i.e. h must in fact be a holomorphic function on the Hopf surface, and so h is a constant. So $f = cW_1/W_2$ is rational in z_1. Swapping the roles of z_1 and z_2 in this argument, f must also be rational in z_2, so a rational function.

Expanding out W_1 and W_2 in Taylor series, the terms $z_1^{k_1} z_2^{k_2}$ in their Taylor series must all scale in the same way: by a factor of $\lambda_1^{k_1} \lambda_2^{k_2}$. If there are no hyperresonances, then there can only be one such term, and it must be the same term in W_1 and W_2, so f is constant. If there is a hyperresonance, we can multiply numerator and denominator Weierstrass polynomials each by a factor of $z_1^{k_1} z_2^{k_2}$ for some integers k_1 and k_2 to arrange that they are both rational functions of u. □

3 Bundles on Hopf surfaces

3.1 Meromorphic sections of line bundles

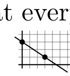
If g is any invertible matrix, say $N \times N$, then we can construct a vector bundle $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,g)} \mathbb{C}^N$ over each Hopf surface S_F by the equivalence $(z, v) \sim (F(z), gv)$. Conjugate linear maps yield isomorphic vector bundles, and splitting g into Jordan blocks yields a sum of vector bundles, so let's assume that g is a single Jordan block. The invariant subspaces of g determine a flag of invariant vector subbundles on the Hopf surface. The meromorphic sections are the solutions of

$$f(F(z)) = g\, f(z).$$

From the exponential sheaf sequence, every line bundle on every Hopf surface has the form $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,a)} \mathbb{C}$ for a unique nonzero complex number a (see [2] p. 226 or [23]).

Proposition 3.1 (Mall [23]). *Take $F : \mathbb{C}^2 \to \mathbb{C}^2$ in the Poincaré domain, and $a \neq 0$ a complex number. The meromorphic sections of the line bundle $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,a)} \mathbb{C}$ are, up to isomorphism (with c an arbitrary complex number, $u = z_1^{m_1}/z_2^{m_2}$, and $P(u)$ and $Q(u)$ arbitrary polynomials):*

$F\left(z_1, z_2\right)$	$a \in \mathbb{C}^\times$	meromorphic sections
$\left(\lambda_1 z_1, \lambda_2 z_2\right), \lambda_1^{m_1} = \lambda_2^{m_2}$	$\lambda_1^{k_1} \lambda_2^{k_2}$	$z_1^{k_1} z_2^{k_2} \frac{P(u)}{Q(u)}$
$\left(\lambda_1 z_1, \lambda_2 z_2\right)$, generic	$\lambda_1^{k_1} \lambda_2^{k_2}$	$c\, z_1^{k_1} z_2^{k_2}$
$\left(\lambda_1 z_1, \lambda_2 z_2\right)$	$a \neq \lambda_1^{k_1} \lambda_2^{k_2}$	0
$\left(\lambda z_1, \lambda^m z_2 + z_1^m\right)$	λ^k	$c\, z_1^k$
$\left(\lambda z_1, \lambda^m z_2 + z_1^m\right)$	$a \neq \lambda^k$	0

Remark 3.2. If F is hyperresonant with hyperresonance (m_1, m_2), and $a = \lambda_1^{k_1} \lambda_2^{k_2}$, then draw a dot at (k_1, k_2) and at every point given by shifting (k_1, k_2) over by integer multiples of $(m_1, -m_2)$:  In particular, the line through these points has negative slope. The Laurent series terms in each meromorphic

section f of $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,a)} \mathbb{C}$ have exponents (j_1, j_2) lying on these points, so each dot represents a meromorphic section, up to scaling. The holomorphic sections of the line bundle arise from the points in the nonnegative quadrant. The tensor product of line bundles is just addition of the points lying on the associated lines.

Proof. Take a meromorphic section, say f. Replacing f with $z_1^{k_1} z_2^{k_2} f$ for various integers k_1 and k_2 gives a meromorphic section of the line bundle with a replaced by $\lambda_1^{k_1} \lambda_2^{k_2} a$. Moreover, $f \mapsto z_1^{k_1} z_2^{k_2} f$ is an isomorphism of meromorphic sections of these line bundles. So we can arrange that f doesn't vanish or have poles at generic points of both axes. But then on each axis, f transforms like $f(\lambda_1 z_1, 0) = a f(z_1, 0)$. Clearly f is meromorphic on both coordinate axes. Taking a Laurent expansion in some annulus around the origin, we find that $f(z_1, 0) = a_1 z_1^{\ell_1}$ for an appropriate choice of integer ℓ_1, and $a = \lambda_1^{\ell_1}$ for some integer ℓ_1. So once again replacing f by some $z_1^{-\ell_1} f$, we can arrange that f is a nonzero constant on the z_1 axis, and that $a = 1$: f is a meromorphic function on the Hopf surface. Applying our classification of meromorphic functions from lemma 2.3 on page 5, finally every meromorphic section of every line bundle $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,a)} \mathbb{C}$ has the form

$$f(z) = z_1^{k_1} z_2^{k_2} \frac{P(u)}{Q(u)} \text{ where } u = z_1^{m_1} / z_2^{m_2}.$$

By cancelling common factors, we can arrange that P and Q have no common zeros and that neither $P(u)$ nor $Q(u)$ have zeros at $u = 0$. $\square$

3.2 Flat bundles of projective lines

We can similarly consider a bundle of projective spaces $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,g)} \mathbb{P}^N$, with $g \in \mathbb{P}\,\mathrm{GL}(N+1, \mathbb{C})$. Such bundles are precisely the flat bundles with projective space fibers over Hopf surfaces. We can assume that g is in Jordan normal form. We will refer to any meromorphic map f on $\mathbb{C}^2 \setminus 0$ valued in $\mathbb{P}^N$ satisfying $f(F(z)) = g\, f(z)$ as a *meromorphic section* of the bundle. For example, if g fixes infinity, we will also equally well allow f to be everywhere infinite rather than being meromorphic, and then also call such a map f a *meromorphic section*. Obviously the constant maps f valued in the locus of fixed points of g will provide meromorphic sections. We will only need to consider flat bundles of projective lines. We will write elements of $\mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$ as matrices in square brackets.

Proposition 3.3. *Suppose that $F : \mathbb{C}^2 \to \mathbb{C}^2$ belongs to the Poincaré domain and $g \in \mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$. The meromorphic sections of the flat projective line bundle $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,g)} \mathbb{P}^1$ are given in table 1 on the following page (after a suitable isomorphism to put F and g into one of the indicated forms). Every meromorphic section is a holomorphic section.*

The proof is split up into several lemmas.

$F(z_1, z_2)$	$g \in \mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$	meromorphic sections
$(\lambda_1 z_1, \lambda_2 z_2)\,, \lambda_1^{m_1} = \lambda_2^{m_2}$	$\begin{bmatrix} \lambda_1^{k_1}\lambda_2^{k_2} & 0 \\ 0 & 1 \end{bmatrix}$	$z_1^{k_1} z_2^{k_2} \frac{P(u)}{Q(u)}, \infty$
$(\lambda_1 z_1, \lambda_2 z_2)\,$, generic	$\begin{bmatrix} \lambda_1^{k_1}\lambda_2^{k_2} & 0 \\ 0 & 1 \end{bmatrix}$	$c\, z_1^{k_1} z_2^{k_2}, \infty$
$(\lambda_1 z_1, \lambda_2 z_2)$	$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \frac{a_1}{a_2} \neq \lambda_1^{k_1}\lambda_2^{k_2}$	$0, \infty$
$(\lambda_1 z_1, \lambda_2 z_2)$	$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$	∞
$(\lambda z_1, \lambda^m z_2 + z_1^m)$	$\begin{bmatrix} \lambda^k & 0 \\ 0 & 1 \end{bmatrix}$	$c\, z_1^k, \infty$
$(\lambda z_1, \lambda^m z_2 + z_1^m)$	$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \frac{a_1}{a_2} \neq \lambda^k$	$0, \infty$
$(\lambda z_1, \lambda^m z_2 + z_1^m)$	$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix},$	$\frac{z_2}{a}\left(\frac{\lambda}{z_1}\right)^m + c, \infty$

Table 1: The meromorphic sections of the flat projective line bundles $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,g)} \mathbb{P}^1$ on Hopf surfaces, with c an arbitrary complex number and $P(u)$ and $Q(u)$ arbitrary polynomials, and $u = z_1^{m_1}/z_2^{m_2}$.

Lemma 3.4. *Take any diagonal linear map $F : \mathbb{C}^2 \to \mathbb{C}^2$, say*

$$F(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$$

in the Poincaré domain and any nondiagonalizable linear fractional transformation $g \in \mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$. Then a meromorphic section of $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,g)} \mathbb{P}^1$ is precisely a constant mapping to the fixed point of g on $\mathbb{P}^1$.

Proof. We can assume that

$$g = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

with $a \neq 0$. Suppose that f is a meromorphic section, which we identify with a meromorphic function on $\mathbb{C}^2$. The poles of $f(0, z_2)$ can't accumulate to 0, unless f is infinite at the generic point of $z_1 = 0$. So either $f = \infty$ on $z_1 = 0$ or else we can take a Laurent series expansion of $f(0, z_2)$ around $z_2 = 0$. Plug in $z_1 = 0$ to see that

$$f(0, \lambda^m z_2) = f(0, z_2) + \frac{1}{a}.$$

The Laurent expansion has inconsistent constant term, unless f is infinite on the line $z_1 = 0$. The same argument swapping the roles of z_1 and z_2 ensures that f is infinite on the line $z_2 = 0$. Therefore f must be infinite at all points of both coordinate axes. Suppose that f is not infinite everywhere. So we can write

$$f = \frac{h}{z_1^{\ell_1} z_2^{\ell_2} W}$$

with $\ell_1, \ell_2 > 0$, where W is a Weierstrass polynomial (for one or the other of the axes) and h is holomorphic, and h and W have no common factors among holomorphic functions. We will pick ℓ_1 and ℓ_2 as large as possible to keep W holomorphic, so W will be finite and nonzero at the generic point of both axes. To be a meromorphic section, we need

$$f(\lambda_1 z_1, \lambda_2 z_2) = f(z_1, z_2) + \frac{1}{a}.$$

In terms of the Weierstrass polynomial,

$$\frac{h(\lambda_1 z_1, \lambda_2 z_2)}{\lambda_1^{\ell_1} \lambda_2^{\ell_2} z_1^{\ell_1} z_2^{\ell_2} W(\lambda_1 z_1, \lambda_2 z_2)} = \frac{h(z_1, z_2) + \frac{1}{a} z_1^{\ell_1} z_2^{\ell_2} W(z_1, z_2)}{z_1^{\ell_1} z_2^{\ell_2} W(z_1, z_2)}.$$

Clearly $z_1^{\ell_1} z_2^{\ell_2} W(\lambda_1 z_1, \lambda_2 z_2)$ is a Weierstrass polynomial (up to a constant factor) for the denominator of the left hand side, while $z_1^{\ell_1} z_2^{\ell_2} W(z_1, z_2)$ is a Weierstrass polynomial for the denominator of the right hand side. By the uniqueness of Weierstrass polynomials, W transforms by scaling, so as a holomorphic section of a holomorphic line bundle, i.e. $W(\lambda_1 z_1, \lambda_2 z_2) = \lambda_1^{k_1} \lambda_2^{k_2} W(z_1, z_2)$ for some integers k_1 and k_2. By remark 3.2, because W is holomorphic, neither k_1 nor k_2 can be negative. Moreover, h must transform according to

$$h(\lambda_1 z_1, \lambda_2 z_2) = \lambda_1^{k_1+\ell_1} \lambda_2^{k_2+\ell_2} \left(h(z_1, z_2) + \frac{1}{a} z_1^{\ell_1} z_2^{\ell_2} W(z_1, z_2) \right).$$

By proposition 3.1 on page 6, W can be written as

$$W(z_1, z_2) = \sum_{j=0}^{N} b_j z_1^{j m_1} z_2^{(N-j) m_2}$$

for some complex numbers $b_0, b_1, \dots, b_N$.

In order that W not vanish on either axis, we must have

$$W(z_1, z_2) = \sum_{s=0}^{N} b_s z_1^{s m_1} z_2^{(N-s) m_2}$$

with $b_0 \neq 0$ and $b_N \neq 0$. In particular, we can take $(k_1, k_2) = (0, N m_2)$. Expanding h into a Taylor series

$$h(z_1, z_2) = \sum_{n_1, n_2} a_{n_1, n_2} z_1^{n_1} z_2^{n_2},$$

and plugging in $(k_1, k_2) = (0, N m_2)$, we see that the term $a_{\ell_1, \ell_2 + N m_2}$ satisfies

$$\lambda_1^{\ell_1} \lambda_2^{\ell_2 + N m_2} a_{\ell_1, \ell_2 + N m_2} = \lambda_1^{\ell_1} \lambda_2^{\ell_2 + N m_2} \left(a_{\ell_1, \ell_2 + N m_2} + \frac{b_0}{a} \right).$$

This forces $b_0 = 0$, a contradiction. Therefore there are no meromorphic sections of such bundles except for $f = \infty$. $\square$

Lemma 3.5. *Take any diagonal linear map $F : \mathbb{C}^2 \to \mathbb{C}^2$, say*

$$F(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$$

in the Poincaré domain and any linear fractional transformation $g \in \mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$. If

$$g = \begin{bmatrix} \lambda_1^{k_1} \lambda_2^{k_2} & 0 \\ 0 & 1 \end{bmatrix}$$

then the flat projective line bundle $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,g)} \mathbb{P}^1$ has meromorphic sections either the constant map $f(z_1, z_2) = \infty$ or

$$f(z_1, z_2) = z_1^{k_1} z_2^{k_2} \frac{P(u)}{Q(u)}$$

where

$$u = z_1^{m_1} / z_2^{m_2}$$

and m_1 and m_2 are from the hyperresonance $\lambda_1^{m_1} = \lambda_2^{m_2}$. (We take $P(u)/Q(u)$ constant if there is no hyperresonance.) If g is not conjugate in $\mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$ to a matrix of the required form above, then a meromorphic section is precisely a constant mapping to one of the fixed points of g on $\mathbb{P}^1$.

Proof. If g is not diagonalizable, then the result is proven in lemma 3.4 on page 8. So assume that g is diagonal, say

$$g = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

and the contraction F is diagonal linear. Then a meromorphic section is a meromorphic function $f : \mathbb{C}^2 \setminus 0 \to \mathbb{C}$ for which $f(\lambda_1 z_1, \lambda_2 z_2) = \frac{a_1}{a_2} f(z_1, z_2)$. Again, such a function f has the form

$$f(z_1, z_2) = z_1^{k_1} z_2^{k_2} \frac{P(u)}{Q(u)} \text{ where } u = z_1^{m_1} / z_2^{m_2}$$

and a_1/a_2 must have the form $\lambda_1^{k_1} \lambda_2^{k_2}$ or else f is constant and equal to 0 or ∞. Therefore

$$g = \begin{bmatrix} \lambda_1^{k_1} \lambda_2^{k_2} & 0 \\ 0 & 1 \end{bmatrix}$$

up to rescaling, or else there are no meromorphic sections other than $f = 0$ and $f = \infty$. ∎

Corollary 3.6. *Every meromorphic section of a flat projective line bundle on any Hopf surface is a holomorphic section.*

Proof. In general, a meromorphic function f on a complex surface need not be a holomorphic map to $\mathbb{P}^1$. It will be a holomorphic map to $\mathbb{P}^1$ just when, near each point, it can be made a holomorphic function by a linear fractional

transformation. Equivalently, either f or $1/f$ is a holomorphic function at each point. Equivalently, either f is finite, or $1/f$ is finite near each point. Equivalently, the zero locus of f does not cross the poles of f. Every meromorphic section of any flat bundle of projective lines over a diagonal Hopf surface is holomorphic, as the zeroes and poles occur only along the curves $z_1 = 0$, $z_2 = 0$ and $z_1^{m_1} = (\text{constant})\, z_2^{m_2}$. □

Lemma 3.7. *Consider an exceptional map F in the Poincaré domain. We can assume that $F(z_1, z_2) = (\lambda z_1, \lambda^m z_2 + z_1^m)$. Take a linear fractional transformation g fixing a single point of $\mathbb{P}^1$. We can assume that*

$$g = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

Then the meromorphic sections of the $\mathbb{P}^1$-bundle $\left(\mathbb{C}^2 \setminus 0\right) \times_{(F,g)} \mathbb{P}^1$ are precisely the functions

$$f = \frac{z_2}{a} \left(\frac{\lambda}{z_1}\right)^m + b,$$

for any constant b. All meromorphic sections are holomorphic.

Proof. Suppose that f is a meromorphic section, which we identify with a meromorphic function on $\mathbb{C}^2$. The poles of $f(0, z_2)$ can't accumulate to 0, unless f is infinite at the generic point of $z_1 = 0$. So either $f = \infty$ on $z_1 = 0$ or else we can take a Laurent series expansion of $f(0, z_2)$ around $z_2 = 0$. Plug in $z_1 = 0$ to see that

$$f(0, \lambda^m z_2) = f(0, z_2) + \frac{1}{a}.$$

The Laurent expansion has inconsistent constant term, unless f is infinite on the line $z_1 = 0$. We can therefore write

$$f(z) = \frac{h(z)}{z_1^\ell W(z)}$$

for some $\ell > 0$, with $h(z)$ holomorphic and $W(z)$ a Weierstrass polynomial in z_1 not dividing into $h(z)$. Then

$$f(F(z)) = \frac{h(F(z))}{\lambda^\ell z_1^\ell W(F(z))} = \frac{h(z) + \frac{1}{a} z_1^\ell W(z)}{z_1^\ell W(z)}.$$

Clearly (up to scaling by a constant) $W(z)$ is a Weierstrass polynomial for the denominator of $z_1^\ell f(F(z))$, as is $W(F(z))$, along the line $z_1 = 0$. By uniqueness of Weierstrass polynomials, $W(F(z)) = c\, W(z)$ for some constant c. But then W must be a holomorphic section of a line bundle on the associated Hopf surface. By proposition 3.1 on page 6, $W(z_1, z_2) = z_1^k$ for some integer $k \geq 0$. So we can assume that

$$f(z) = \frac{h(z)}{z_1^\ell}$$

with $\ell > 0$. Therefore

$$h\left(\lambda z_1, \lambda^m z_2 + z_1^m\right) = \lambda^\ell \left(h\left(z_1, z_2\right) + \frac{z_1^\ell}{a}\right).$$

Again restrict to $z_1 = 0$ to see that

$$h\left(0, \lambda^m z_2\right) = \lambda^\ell h\left(0, z_2\right).$$

Expand h in a Taylor series to see that $\ell = km$ for some integer $k \geq 0$ and $h\left(0, z_2\right) = \frac{z_2^k}{a}$. So we see that

$$h\left(\lambda z_1, \lambda^m z_2 + z_1^m\right) = \lambda^{km} \left(h\left(z_1, z_2\right) + \frac{z_1^{km}}{a}\right).$$

Following Kodaira [18] p. 697 equation 100, we let $h_1 = \frac{\partial h}{\partial z_2}$. Then we calculate that

$$h_1\left(F(z)\right) = \lambda^{(k-1)m} h_1(z).$$

Therefore h_1 is a holomorphic section of a line bundle, and by proposition 3.1 on page 6,

$$h_1\left(z_1, z_2\right) = c\, z_1^{(k-1)m},$$

so that

$$h\left(z_1, z_2\right) = c\, z_1^{(k-1)m} z_2 + \sum_{s=0}^{\infty} a_s z_1^s,$$

for some complex numbers a_s, and we plug in to find

$$\begin{aligned} 0 &= h\left(\lambda z_1, \lambda^m z_2 + z_1^m\right) - \lambda^{km} \left(h\left(z_1, z_2\right) + \frac{z_1^{km}}{a}\right) \\ &= \sum_{s=0}^{m} \left(\left(\lambda^s - \lambda^{km}\right) a_s + \delta_{s=km} \left(c - \frac{\lambda^m}{a}\right) \lambda^{(k-1)m}\right) z_1^s. \end{aligned}$$

Looking at the z_1^{km} coefficient yields $c = \lambda^m / a$. Plugging in $s \neq km$ yields $a_s = 0$. Therefore

$$h\left(z_1, z_2\right) = \frac{\lambda^m\, z_1^{(k-1)m} z_2}{a} + b\, z_1^{km}.$$

Finally

$$f\left(z_1, z_2\right) = \frac{\lambda^m z_2}{a z_1^m} + b.$$

□

4 Geometric structures on Hopf surfaces

4.1 Geometric structures

If G/H is any homogeneous space, a G/H-structure is a maximal atlas of charts valued in G/H, with transition maps in G. The identity map of a homogeneous space G/H is contained in a unique G/H-structure, and is called the *model* G/H-structure. If G/H is affine (projective) space and G is the group of affine (projective) transformations, then G/H-structures are called *affine (projective) structures*. Clearly any linear Hopf surface S_F bears an affine structure, since the transition map F is linear.

Example 4.1. For example, let $G = \mathrm{GL}(2, \mathbb{C})$ and H the subgroup fixing the point $(1, 0) \in \mathbb{C}^2$. So $G/H = \mathbb{C}^2 \setminus 0$. Every linear Hopf surface has an obvious G/H-structure, since its universal covering space is G/H and its covering group acts by an element of G.

Example 4.2. If $G = \mathbb{P}\,\mathrm{GL}(n+1, \mathbb{C})$ and H is the stabilizer of a point of $\mathbb{P}^n$, then a G/H-structure is called a *projective connection*.

4.2 Developing maps and holonomy

Lemma 4.3. *Every G/H-structure on any manifold M is obtained from a local diffeomorphism* $\mathrm{dev} : \tilde{M} \to G/H$ *of the universal covering space of M (called the* developing map*), equivariant for a homomorphism* $\mathrm{hol} : \pi_1(M) \to G$ *(called the* holonomy*): to recover the G/H-structure, compose* dev *with a local inverse of $\tilde{M} \to M$ to give an atlas of local coordinates valued in G/H.*

The pair (dev, hol) *are only defined up to the G-action* $\left(g\ \mathrm{dev}, g\ \mathrm{hol}\ g^{-1}\right)$. *Conversely, the G-orbit of this pair under this action determines the G/H-structure. Moreover, any choice of two maps* $\mathrm{dev} : \tilde{M} \to G/H$ *and* $\mathrm{hol} : \pi_1(M) \to G$ *with* hol *a group morphism and* dev *a* hol*-equivariant local diffeomorphism determines a unique G/H-structure on M.*

Proof. See Thurston [30] p. 140. ☐

Therefore we will classify G/H-structures on Hopf surfaces by writing out their developing maps and holonomies.

Definition 4.4. If M is a complex manifold and G is a complex Lie group with H a closed complex subgroup, then a G/H-structure is called *holomorphic* if its developing map is holomorphic, or equivalently if all of the charts of the structure are holomorphic maps. From now on all locally homogeneous structures will be assumed holomorphic.

Definition 4.5. A structure is called *complete* if the developing map is onto.

Definition 4.6. A structure is called *essential* if the developing map is injective.

Essential structures are precisely the induced structures on manifolds covered by open sets of the model. For example, a Riemann surface of any genus has the obvious holomorphic projective connection given by the inclusion $\Delta \subset \mathbb{C} \subset \mathbb{P}^1$.

Definition 4.7. On a Hopf surface S_F, the fundamental group has the distinguished generator F, so the holonomy map hol is determined by the element $\mathrm{hol}(F) \in G$; refer to this element as the *holonomy generator* of the G/H-structure.

Definition 4.8. A *branched G/H-structure* on a manifold M is a choice of map $\tilde{M} \to G/H$ (again called the *developing map*), equivariant for a homomorphism $\pi_1(M) \to G$ (again called the *holonomy*), determined up to the same G-action.

The developing map of a branched structure might *not* be a local biholomorphism. The basic difficulty we encounter in this paper is that of distinguishing branched from unbranched structures. Obstructions to structures are usually also obstructions to branched structures, so if there is a branched structure, then most of the obstructions we can come up with will not help us to rule out the possibility of an unbranched structure.

4.3 Induction of structures

Definition 4.9. Suppose that G_0/H_0 and G/H are homogeneous spaces. A *morphism* of homogeneous spaces $\Phi : G_0/H_0 \to G/H$ means a morphism $\Phi : G_0 \to G$ of Lie groups so that $\Phi(H_0) \subset H$. We will also denote the induced map $G_0/H_0 \to G/H$ by the letter Φ. A morphism of homogeneous spaces is called an *avatar* if the induced smooth map $G_0/H_0 \to G/H$ is a local diffeomorphism.

Definition 4.10. If $\Phi : G_0/H_0 \to G/H$ is an avatar, and we have a G_0/H_0-structure on a manifold M, with developing map $\mathrm{dev}_0 : \tilde{M} \to G_0/H_0$ and holonomy $\mathrm{hol}_0 : \pi_1(M) \to G_0$, then the *induced* G/H-structure is the one given by $\mathrm{dev} = \Phi \circ \mathrm{dev}_0$ and $\mathrm{hol} = \Phi \circ \mathrm{hol}_0$.

5 The model

5.1 Definition

As usual, we treat points of $\mathbb{P}^1$ as lines through 0 in $\mathbb{C}^2$, and we write $\mathcal{O}(n)$ for the bundle over $\mathbb{P}^1$ whose fiber over a point $L \in \mathbb{P}^1$ is the n-fold symmetric product $\mathrm{Sym}^n(L)^*$, if $n > 0$, and $\mathrm{Sym}^{|n|}(L)$ if $n < 0$, and $\mathbb{C}$ if $n = 0$. We will denote the total space of the bundle $\mathcal{O}(n)$ also as $\mathcal{O}(n)$. Clearly $\mathcal{O}(n)$ is a complex surface. For now, let's assume that $n > 0$ and write the points of $\mathcal{O}(n)$ as pairs (L, q) with $q \in \mathrm{Sym}^n(L)^*$. Thus the global sections of $\mathcal{O}(n) \to \mathbb{P}^1$ are the homogeneous polynomials of degree n, $\mathrm{Sym}^n\left(\mathbb{C}^2\right)^*$. Let

$$G = (\mathrm{GL}(2, \mathbb{C})/n\text{-th roots of }1) \rtimes \mathrm{Sym}^n\left(\mathbb{C}^2\right)^*$$

act on $\mathcal{O}(n)$ by $(g, p)(L, q) = \left(gL, q\,g^{-1} + p|_{gL}\right)$. The multiplication in G is $(g_0, p_0)(g_1, p_1) = \left(g_0 g_1, p_0 + p_1 g_0^{-1}\right)$ and the inverse operation is $(g, p)^{-1} = \left(g^{-1}, -pg\right)$. Let H be the stabilizer of $(L_0, 0)$, where L_0 is the line $z_2 = 0$ in

$\mathbb{C}^2$; i.e.

$$H = \left\{ (g,p) \in G \,\middle|\, g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},\ p(1,0) = 0 \right\}.$$

Clearly $\mathcal{O}(n) = G/H$. Moreover, G acts freely on $\mathcal{O}(n)$. The action of G preserves the fiber bundle map $\mathcal{O}(n) \to \mathbb{P}^1$, and preserves the affine structure on each fiber. It also acts transitively on the global sections of $\mathcal{O}(n) \to \mathbb{P}^1$, a family of curves transverse to the fibers. Moreover it acts on $\mathbb{P}^1$ via a surjection to the group of linear fractional transformations.

Every surface with $\mathcal{O}(n)$-structure inherits a foliation, corresponding to the fiber bundle map, and inherits a family of curves which locally are identified with the global sections. Locally, on open sets on which the foliation is a fibration, the base space of the fibration is a Riemann surface with projective structure; in this sense the foliation has a *transverse projective structure*.

5.2 Ordinary differential equations and geometric structures

It is well known (see Lagrange [21, 22], Fels [6, 7], Dunajski and Tod [5], Godliński and Nurowski [9], Doubrov [4]) that every (real or holomorphic) scalar ordinary differential equation of order $n+1 \geq 3$ has a symmetry Lie algebra of point transformations of dimension at most $n+5$, and this dimension is acheived just precisely for the ordinary differential equations which are locally identified by point transformation with the equation $\frac{d^{n+1}y}{dx^{n+1}} = 0$. Moreover, every holomorphic scalar ordinary differential equation locally point equivalent to $\frac{d^{n+1}y}{dx^{n+1}} = 0$ is locally determined by, and locally determines, an $\mathcal{O}(n)$-structure. Each solution of the differential equation is identified by the developing map with a global section of $\mathcal{O}(n)$.

5.3 Conjugacy classes in the symmetry group

Recall that

$$G = \left(\mathrm{GL}(2,\mathbb{C}) / n\text{-th roots of } 1\right) \rtimes \mathrm{Sym}^n \left(\mathbb{C}^2\right)^*.$$

In this section we show that every element $(g,p) \in G$ is conjugate to one of a certain normal form defined in definition 5.8 on page 18.

The conjugates of an element $(g,p) \in G$ are the elements of the form

$$\left(g_0 g g_0^{-1}, p g_0^{-1} + p_0 - p_0 g_0 g^{-1} g_0^{-1}\right).$$

Lemma 5.1. *Pick a matrix $g \in \mathrm{GL}(2,\mathbb{C})$. There are no nonzero g-invariant homogeneous polynomials of degree n just when, for all homogeneous polynomials p of degree n, (g,p) is conjugate to $(g,0)$.*

Proof. We can take $g_0 = I$, and then we have to solve $p_0 g^{-1} - p_0 = p$. The kernel of the map $p_0 \mapsto p_0 g^{-1} - p_0$ is precisely the g invariant homogeneous polynomials of degree n. Therefore the linear map $p_0 \mapsto p_0 g^{-1} - p_0$ is onto just when there are no g-invariant polynomials. □

Corollary 5.2. *If $g \in \mathrm{GL}(2, \mathbb{C})$ lies in the Poincaré domain, or if g^{-1} does, then, for any homogeneous polynomial p of any positive degree, (g, p) is conjugate to $(g, 0)$*

Lemma 5.3. *If g is not diagonalizable, then, for any homogeneous polynomial p of any positive degree, either (1) (g, p) is conjugate to $(g, 0)$ or (2) (g, p) is conjugate to*

$$\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, Z_1^n \right).$$

Proof. Suppose that

$$g = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

and that (g, p) is not conjugate to $(g, 0)$. By lemma 5.1 on the preceding page, we can take a nonzero g-invariant polynomial p_0 of degree n. Factor out as many factors of z_2 from $p_0(z_1, z_2)$ as possible, say

$$p_0(z_1, z_2) = q_0(z_1, z_2)\, z_2^k.$$

Then q_0 must scale under g-action by $\frac{1}{a^k}$. Suppose that q_0 has degree m. Write out coefficients

$$q_0(z_1, z_2) = \sum b_i z_1^i z_2^{m-i}.$$

The highest order term in z_1 must be $b_n z_1^m \neq 0$, since otherwise we could factor out more factors of z_2 from q_0. Compute out

$$q_0(gz) = b_m a^m z_1^m + \left(m b_m a^{m-1} + b_{m-1} a^m \right) z_1^{m-1} z_2 + \dots$$

In order that q_0 scale by $\frac{1}{a^k}$, we must have

$$\begin{aligned} b_m a^m &= \frac{b_m}{a^k} \\ m b_m a^{m-1} + b_{m-1} a^m &= \frac{b_{m-1}}{a^k}. \end{aligned}$$

Since $b_m \neq 0$, we must have $a^{m+k} = 1$, a root of unity. But then the second equation becomes $m b_m = 0$. Since $b_m \neq 0$, we must have $m = 0$, so q_0 is constant and $p_0 = c\, z_2^n$.

Because p_0 is g-invariant, we must have $a^n = 1$. Since $(g, p) \in G$ has matrix part g defined only up to multiplication by n-th roots of 1, we can arrange $a = 1$. The polynomials that we can arrive at in the form $p_0 - p_0 g^{-1}$ are clearly precisely those of the form

$$\sum_{j=1}^n b_j \left(\binom{j}{1} z_1^{j-1} z_2^{n-j+1} - \binom{j}{2} z_1^{j-2} z_2^{n-j+2} + \dots + (-1)^j \binom{j}{j-1} z_1 z_2^{n-1} + z_2^n \right).$$

Looking at the leading terms in z_1, we see that we can successively pick $b_1, b_2, \dots$ to kill off the $z_1^{n-1} z_2, z_1^{n-2} z_2^2, \dots$ terms in p by conjugation by (I, p_0), until we kill off all terms except the z_1^n term. Then we rescale by conjugation by $(\lambda I, 0)$ to rescale p as needed to arrange $p = z_1^n$. $\square$

Definition 5.4. A matrix $g \in \mathrm{GL}(2, \mathbb{C})$ is called *hyperresonant* if it is diagonalizable with eigenvalues λ_1, λ_2 satisfying $\lambda_1^{m_1} = \lambda_2^{m_2}$ for some pair of integers $(m_1, m_2) \neq (0, 0)$. Such a pair of integers (m_1, m_2) will be called a *hyperresonance pair* of g, and the collection of hyperresonance pairs (an abelian subgroup of $\mathbb{Z}^2$) will be called the *hyperresonance group* Λ_g of g. If the hyperresonance group is of rank 1, we take the element (m_1, m_2) with smallest positive m_1 (or $(0, m_2)$ with smallest positive m_2), and call it the *hyperresonance* of g.

Lemma 5.5. *The hyperresonance group of a diagonalizable matrix g with eigenvalues λ_1, λ_2 has rank 0 just when g is not hyperresonant, rank 2 just when $\lambda_1 = e^{2\pi i p_1/q_1}$ and $\lambda_2 = e^{2\pi i p_2/q_2}$ where $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are rational numbers, and rank 1 otherwise.*

Proof. Suppose that $\lambda_1 = e^{2\pi i p_1/q_1}$ and $\lambda_2 = e^{2\pi i p_2/q_2}$ where $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are rational numbers. Clearly $(q_1, 0)$ and $(0, q_2)$ lie in Λ_g, so Λ_g has rank 2.

Conversely, suppose that the rank of Λ_g is 2. Suppose that $\lambda_1^{m_1} = \lambda_2^{m_2}$ with $(m_1, m_2) \neq (0, 0)$. Let $r_j = \log |\lambda_j|$. Then $m_1 r_1 = m_2 r_2$. So the hyperresonance group lies on a line through 0 in $\mathbb{R}^2$, unless $r_1 = r_2 = 0$. If $r_1 \neq 0$ or $r_2 \neq 0$, then there is an integer point on that line with smallest nonzero distance from the origin, and Λ_g is of rank 1.

So we can suppose that $r_1 = r_2 = 0$, i.e. $\lambda_1 = e^{2\pi i a_1}$ $\lambda_2 = e^{2\pi i a_2}$ for some real numbers $0 \leq a_1, a_2 < 1$. The hyperresonance pairs are just the pairs of integers (m_1, m_2) for which $m_1 a_1 + m_1 a_2$ is an integer. The hyperresonance group spans $\mathbb{R}^2$, since it doesn't lie on a line. Take two linearly independent hyperresonances (m_1, m_2) and (n_1, n_2). Then

$$\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

where b_1, b_2 are integers. Therefore

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

are rational numbers, say $a_j = p_j/q_j$ with p_j, q_j integers, $j = 1, 2$. $\square$

Definition 5.6. Suppose that $(g, p) \in G$ and that g is diagonalizable. The *resonant degrees* of p are the integers k with $0 \leq k \leq n$ for which the eigenvalues λ_1, λ_2 of g satisfy $\lambda_1^k \lambda_2^{n-k} = 1$. If we write

$$p(Z_1, Z_2) = \sum_{k=0}^{n} a_k Z_1^k Z_2^{n-k},$$

the *resonant terms* of p are the terms $a_k Z_1^k Z_2^{n-k}$ for which k is a resonant degree. The *leading resonant term* is the term $a_k z_1^k z_2^{n-k}$ with smallest resonant degree k for which $a_k \neq 0$. The *trailing resonant term* is the term $a_k z_1^k z_2^{n-k}$ with largest resonant degree k for which $a_k \neq 0$.

Definition 5.7. Suppose that $(g, p) \in G$ and that g is not diagonalizable. We will declare that n is a *resonant degree* of p if g has eigenvalue λ an n-th root of 1, and declare that there are no resonant terms otherwise. Once again, the resonant terms are the nonzero terms of resonant degree.

Definition 5.8. We will say that an element $(g, p) \in G$ is in *normal form* if either

1. $$g = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

   and $p = 0$ or

2. $$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

   and $p(Z_1, Z_2) = Z_1^n$ or

3. $$g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

   is diagonal and

   $$p(Z_1, Z_2) = \sum_{k=0}^{n} a_k Z_1^k Z_2^{n-k}$$

   with all nonresonant terms vanishing, and $a_k = 1$ for both the leading and trailing resonant terms.

Lemma 5.9. *Every element $(g, p) \in G$ is conjugate to an element in normal form. Either (1) the normal form is unique up to possibly permuting coordinates z_1 and z_2 or (2) $g = I$ and all terms of p are resonant.*

Suppose that there are at least two distinct resonant terms. Then (1) g is diagonalizable with eigenvalues $\lambda_1 = e^{2\pi i p_1/q_1}$ and $\lambda_2 = e^{2\pi i p_2/q_2}$ with $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ rational numbers and (2) for each resonant degree k,

$$k\frac{p_1}{q_1} + (n-k)\frac{p_2}{q_2}$$

is an integer.

Proof. By lemma 5.3 on page 16, we can assume that

$$g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Suppose that

$$p(Z_1, Z_2) = \sum_k a_k Z_1^k Z_2^{n-k}.$$

Then pick any element $(g_0, p_0) \in G$ of the form

$$g_0 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

and say

$$p_0 (Z_1, Z_2) = \sum_k b_k Z_1^k Z_2^{n-k}.$$

Define a polynomial p_1 by

$$(g, p_1) = (g_0, p_0)(g, p)(g_0, p_0)^{-1}.$$

i.e.

$$p_1 = p g_0^{-1} + p_0 - p_0 g^{-1}.$$

Therefore the coefficients of p_1 are

$$\frac{a_k}{\mu_1^k \mu_2^{n-k}} + b_k \left(1 - \frac{1}{\lambda_1^k \lambda_2^{n-k}}\right).$$

So we can conjugate (g, p) to arrange $a_k = 0$ by choice of b_k unless k is a resonance degree. So we can and will assume that p has only resonant terms. If there is exactly one resonant term, say degree k, then we can arrange by choice of the coefficients μ_1 and μ_2 that $p(Z_1, Z_2) = Z_1^k Z_2^{n-k}$.

If p has two or more resonant terms, then we can find two corresponding resonant degrees, say k_1 and k_2. The hyperresonant pairs $(k_1, n - k_1)$ and $(k_2, n - k_2)$ are linearly independent elements of the hyperresonance group, which must therefore have rank 2. By lemma 5.5 on page 17, g has eigenvalues $\lambda_1 = e^{2\pi i p_1/q_1}$ and $\lambda_2 = e^{2\pi i p_2/q_2}$ with $\frac{p_1}{q_1}, \frac{p_2}{q_2}$ rational numbers, and

$$k \frac{p_1}{q_1} + (n - k) \frac{p_2}{q_2}$$

is an integer for each resonant degree k.

Next we can pick μ_1 and μ_2 to arrange that the leading and trailing resonant coefficients are 1. So we have acheived normal form. If the eigenvalues λ_1 and λ_2 are not equal, then the only choices of g_0 which will preserve the diagonalization of g are diagonal themselves, and we immediately see that there is a unique normal form up to swapping the coordinates Z_1 and Z_2.

If the normal form is not unique, then $g = \lambda I$, for some constant $\lambda \in \mathbb{C}^\times$. If λ is not an n^{th} root of 1, then there are no g invariant homogeneous polynomials, so we can arrange after conjugation $(g, p) = (\lambda I, 0)$, arriving at normal form. Suppose that λ is an n^{th} root of 1. But $(\lambda I, p) = (I, p) \in G$, since we mod out by n^{th} roots of 1. So we can assume that $\lambda = 1$, i.e. $g = I$. Under conjugation, p is acted on by g_0. The roots of p, with multiplicities, are transformed by linear fractional transformation. Since g_0 can rescale p, the conjugacy classes of elements of G of the form (I, p) are precisely identified with choices of n unordered points on $\mathbb{P}^1$, not necessarily distinct, modulo linear fractional transformations of $\mathbb{P}^1$. For more on these conjugacy classes, see Popov and Vinberg [27] p. 140 or Howard et. al. [13]. $\square$

Definition 5.10. We will say that an element $(g, p) \in G$ is *generic* if it is conjugate to an element of the form $(g', 0)$ where

$$g' = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

is diagonal (with the same eigenvalues as g).

Generic elements form a dense open subset of G. An element (g, p) is not generic just when (1) g is not diagonalizable or (2) g is hyperresonant and p has at least one resonant degree.

5.4 Affine coordinates

We want to cover $\mathcal{O}(n)$ in coordinate charts, which we will refer to as *affine coordinates* on $\mathcal{O}(n)$. Take a line L_0 in $\mathbb{C}^2$, which we will also think of as a point of $\mathbb{P}^1$. Consider the open subset of $\mathcal{O}(n)$ which lies over $\mathbb{C} = \mathbb{P}^1 \setminus L_0$. This open subset of $\mathcal{O}(n)$ is preserved by all of the elements $(g, p) \in G$ for which g fixes the line L_0. Let's use linear coordinates Z_1, Z_2 on $\mathbb{C}^2$. Take L_0 to be the line $Z_2 = 0$. We will now produce coordinates t_1, t_2 on the corresponding open subset of $\mathcal{O}(n)$. Map $(t_1, t_2) \in \mathbb{C}^2 \mapsto (L, q) \in \mathcal{O}(n)$ where L is the line $Z_1 = t_1 Z_2$, and $q = t_2 \, Z_2^n|_L$. Clearly $t_1 = Z_1/Z_2$ is an affine chart on $\mathbb{P}^1$. In these coordinates, if we let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the element $(g, 0) \in G$ acts by

$$(g, 0)(t_1, t_2) = \left(\frac{a t_1 + b}{c t_1 + d}, \frac{t_2}{(c t_1 + d)^n} \right).$$

If $p(Z_1, Z_2) = \sum_{i+j=n} a_{ij} Z_1^i Z_2^j$, then the element $(I, p) \in G$ acts by

$$\begin{aligned} (I, p)(t_1, t_2) &= (t_1, t_2 + p(t_1, 1)) \\ &= \left(t_1, t_2 + \sum_{i+j=n} a_{ij} t_1^i \right). \end{aligned}$$

We cover $\mathcal{O}(n)$ in two coordinate charts: (t_1, t_2) and

$$(s_1, s_2) = \left(\frac{1}{t_1}, \frac{t_2}{t_1^n} \right) = g(t_1, t_2).$$

where

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The global sections $p(Z_1, Z_2)$ of $\mathcal{O}(n)$, i.e. homogeneous polynomials of degree n, when written in these coordinates become $t_2 = p(t_1, 1)$. In particular, they satisfy

$$\frac{d^{n+1} t_2}{d t_1^{n+1}} = 0.$$

$\mathcal{O}(n)$-structure	F	developing map	holonomy generator
radial	linear	$\left(\frac{z_1}{z_2}, \frac{1}{z_2^n}\right)$	$(F, 0)$
eigenstructure	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	(z_1, z_2)	$\left(\begin{pmatrix} \frac{\lambda_1}{\lambda_2^{1/n}} & 0 \\ 0 & \frac{1}{\lambda_2^{1/n}} \end{pmatrix}, 0\right)$
eigenstructure	$(\lambda z_1, \lambda^m z_2 + z_1^m)$	(z_1, z_2)	$\left(\begin{pmatrix} \frac{\lambda}{\lambda^{m/n}} & 0 \\ 0 & \frac{1}{\lambda^{m/n}} \end{pmatrix}, \frac{1}{\lambda^m} Z_1^m Z_2^{n-m}\right)$
hyperresonant	see table 3	see table 3	see table 3

Table 2: Examples of structures on each Hopf surface S_F, expressed in affine coordinates.

6 Examples

Our examples are summarized in tables 2 and 3 on page 24. We will now explain them in detail.

6.1 The radial structures on linear Hopf surfaces

We can map $\mathcal{O}(n) \to \mathcal{O}(kn)$ by $(L, q) \mapsto \left(L, q^k\right)$ for any integer $k > 0$. Similarly, we can map $\mathcal{O}(n) \setminus 0 \to \mathcal{O}(kn) \setminus 0$ for all integer values of k: if $k < 0$ then take any element (L, q) with $q \neq 0$ to the pair $(L, r^{|k|})$ where r is dual to q in $\operatorname{Sym}^n(L)$. These maps are local biholomorphisms away from the 0-sections. Moreover, these maps are equivariant under $\mathrm{GL}(2, \mathbb{C})$. In particular, $\mathcal{O}(-1) \setminus 0 = \mathbb{C}^2 \setminus 0$ maps by local biholomorphism to $\mathcal{O}(n)$ for every $n \neq 0$. The $\mathcal{O}(n)$-structures induced by this map on $\mathbb{C}^2 \setminus 0$ are invariant under linear isomorphisms of $\mathbb{C}^2$. Therefore they quotient to every linear Hopf surface. We will refer to these $\mathcal{O}(n)$-structures as the *radial* $\mathcal{O}(n)$-structures on Hopf surfaces. In affine coordinates, each radial structure has developing map

$$(z_1, z_2) \mapsto (t_1, t_2) = \left(\frac{z_1}{z_2}, \frac{1}{z_2^n}\right)$$

defined where $z_2 \neq 0$. Where $z_2 = 0$, we can just swap indices of z_1 and z_2 to get another affine chart, so we can see that the structure is holomorphic. The image of the developing map is the complement of the 0-section in $\mathcal{O}(n)$, so the structure is incomplete. The developing map is an n-fold covering of its image, so is inessential. The holonomy generator is $(g, p) = (F, 0)$. More generally, swapping indices of z_1 and z_2 is an involution on the space of $\mathcal{O}(n)$-structures on diagonal Hopf surfaces.

6.2 The eigenstructures on linear Hopf surfaces

A related example: consider $\mathbb{C}^2$ foliated by vertical lines, i.e. the lines $z_1 =$ constant. The affine transformations of $\mathbb{C}^2$ which preserve this foliation are

precisely the maps of the form

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

with $a_{11} \neq 0$ and $a_{22} \neq 0$. Let G_1 be the group of all such affine transformations, and H_1 the subgroup fixing the origin, i.e. with $b_1 = b_2 = 0$. Clearly the graph of any polynomial function $z_2 = z_2(z_1)$ of degree n is carried to the graph of another polynomial of the same degree by any element of G_1. It follows (as we will shortly see) that there is an invariant $\mathcal{O}(n)$-structure for which these graphs correspond to the global sections of $\mathcal{O}(n) \to \mathbb{P}^1$, and the vertical lines to the fibers.

Consider the subgroup $G_0 \subset G$ consisting of elements $(g, p) \in G$ of the form

$$g = \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}, \ p(Z_1, Z_2) = c_0 Z_2^n + c_1 Z_1 Z_2^{n-1}.$$

Of course, g is defined as a matrix only up to scaling by n-th roots of 1. Consider the complex Lie group isomorphism

$$(g, p) \in G_0 \mapsto (a, b) \in G_1$$

given by

$$\begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{g_{11}}{g_{22}} & 0 \\ \frac{g_{11}}{g_{22}} c_1 & \frac{1}{g_{22}^n} \end{pmatrix}, \ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{g_{12}}{g_{22}} \\ c_0 + \frac{g_{11}}{g_{22}} c_1 \end{pmatrix}.$$

This isomorphism identifies G_1 with the subgroup $G_0 \subset G$, and H_1 with $H_0 \subset H$. Therefore a G/H-structure is induced by a G_1/H_1-structure, i.e. an affine structure foliated by parallel complex geodesics, if and only if the holonomy of the G/H-structure lies in the subgroup G_1 and the developing map has image in $\mathcal{O}(n)$ lying inside the open orbit of G_0.

Another related example: suppose that S is a complex surface with a complex affine structure, foliated by parallel geodesics. Locally we can construct coordinates (t_1, t_2) on S which identify open sets of S with open sets of $\mathbb{C}^2$, and identify the parallel geodesic foliation with the foliation of $\mathbb{C}^2$ by vertical lines. Moreover, the transition maps will now preserve the vertical direction, and therefore are compositions of (1) translations, (2) rescalings of horizontal and vertical axes, and (3) addition of a linear function of t_1 to t_2. In particular, any graph of a polynomial function $t_2 = t_2(t_1)$ will remain a graph of a polynomial of the same degree. In the standard flat affine structure on the torus, there is a foliation by parallel geodesics in each direction, and associated to each such foliation is a translation invariant $\mathcal{O}(n)$-structure for every n.

Let's return now to Hopf surfaces. Pick an eigenline of a linear Poincaré domain map F. The affine lines parallel to that line form an F-invariant foliation of $\mathbb{C}^2$. The associated $\mathcal{O}(n)$-structure descends to the associated Hopf surface. Let's call this the $\mathcal{O}(n)$-*eigenstructure*; there are two such for F with

two distinct eigenvalues, one for F not diagonalizable, and infinitely many for F a homothety. Up to isomorphism of the Hopf surface and perhaps a permutation of indices, the eigenline can be arranged to be the vertical axis, with contraction map of the form $F(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$. Then the developing map of the eigenstructure is given by id : $(z_1, z_2) \mapsto (t_1 = z_1, t_2 = z_2)$, and the holonomy generator is $(g, 0)$ where

$$g = \begin{pmatrix} \frac{\lambda_1}{\lambda_2^{1/n}} & 0 \\ 0 & \frac{1}{\lambda_2^{1/n}} \end{pmatrix}.$$

The image misses precisely the fiber of $\mathcal{O}(n) \to \mathbb{P}^1$ over the point $\infty \in \mathbb{P}^1$, and the origin of the fiber of $\mathcal{O}(n) \to \mathbb{P}^1$ over the point $0 \in \mathbb{P}^1$. The structure is incomplete but is essential.

Another example: take any compact curve C of genus $g \geq 0$, and equip C with a projective structure. (Every curve admits a projective structure; see [11].) The holonomy morphism $\pi_1(C) \to \mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$ lifts to a morphism $\pi_1(C) \to \mathrm{SL}(2, \mathbb{C})$ (see Gallo, Kapovich and Marden [8]). The developing map $\tilde{C} \to \mathbb{P}^1$ of the projective structure pulls back the $\mathcal{O}(n)$-bundle to a line bundle over C. Cut out the zero section of this line bundle and quotient the fibers by any homothety $w \to aw$ with $a \neq 0$. The result is an $\mathcal{O}(n)$-structure on a principal fibration by elliptic curves over the curve of genus g. The developing maps of projective structures of curves of large genus can be very complicated (see Gallo, Kapovitch and Marden [8]), so the developing map of the $\mathcal{O}(n)$-structure cannot be made explicit. The holonomy morphism $\pi_1(C) \rtimes \mathbb{Z} \to G$ takes the generator of $\mathbb{Z}$ to the homothety, and $\pi_1(C) \to \mathrm{GL}(2, \mathbb{C}) / (n\text{-th roots of } 1)$ is the lift of the holonomy morphism of the projective structure on C. The structure is incomplete, but is essential.

6.3 The eigenstructures on exceptional Hopf surfaces

Pick any integers $0 < m \leq n$. Let $(g, p) \in G$ be the element

$$g = \begin{pmatrix} \frac{\lambda}{\varepsilon} & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}, \; p(Z_1, Z_2) = \frac{1}{\lambda^m} Z_1^m Z_2^{n-m}$$

where ε is any solution of $\varepsilon^n = \lambda^m$. (We obtain the same element $(g, p) \in G$ for any choice of ε.) This element (g, p) acts on $\mathcal{O}(n)$ via

$$(g, p)(t_1, t_2) = \left(\lambda t_1, \lambda^m t_2 + t_1^m\right).$$

In particular, every exceptional Hopf surface of degree m has an $\mathcal{O}(n)$-structure, for all $n \geq m$, which we call the *eigenstructure* on the exceptional Hopf surface. The holonomy generator is (g, p), and the developing map is id : $(z_1, z_2) \mapsto (t_1 = z_1, t_2 = z_2)$. The image misses precisely the fiber of $\mathcal{O}(n) \to \mathbb{P}^1$ over the point $\infty \in \mathbb{P}^1$, and the origin of the fiber of $\mathcal{O}(n) \to \mathbb{P}^1$ over the point $0 \in \mathbb{P}^1$. In particular, the structure is incomplete, but is essential.

condition	developing map	holonomy generator
$m_2 = n\,m_1, m_1 \geq 2$ or $n \geq 2$	$\left(\frac{z_1^{m_1} - a_1\,z_2^{m_2}}{z_2}, \frac{z_1}{z_2^n}\right)$	$\begin{pmatrix} \frac{\lambda_1^{m_1}}{\lambda_1^{1/n}} & 0 \\ 0 & \frac{\lambda_2}{\lambda_1^{1/n}} \end{pmatrix}$
$m_2 = n\,m_1, N \geq 2$	$\left(\frac{\prod_{j=1}^N \left(z_1^{m_1} - a_j\,z_2^{m_2}\right)}{z_2}, \frac{z_1}{z_2^n}\right)$	$\begin{pmatrix} \frac{\lambda_1^{m_1 N}}{\lambda_1^{1/n}} & 0 \\ 0 & \frac{\lambda_2}{\lambda_1^{1/n}} \end{pmatrix}$
$m_1 = m_2, m_1 N \neq n$	$\left(\frac{z_1}{z_2}, \frac{\prod_{j=1}^N \left(z_1^{m_1} - a_j\,z_2^{m_2}\right)}{z_2^n}\right)$	$\begin{pmatrix} \frac{\lambda_1}{\lambda_2^{m_2 N/n}} & 0 \\ 0 & \frac{\lambda_2}{\lambda_2^{m_2 N/n}} \end{pmatrix}$
$m_1 = n\,m_2, m_2 \geq 2$ or $n \geq 2$	$\left(\frac{z_1}{z_1^{m_1} - a_1\,z_2^{m_2}}, \frac{z_2}{\left(z_1^{m_1} - a_1\,z_2^{m_2}\right)^n}\right)$	$\begin{pmatrix} \frac{\lambda_1}{\lambda_2^{1/n}} & 0 \\ 0 & \frac{\lambda_1^{m_1}}{\lambda_2^{1/n}} \end{pmatrix}$
$m_1 = n\,m_2, N \geq 2$	$\left(\frac{z_1}{\prod_{j=1}^N \left(z_1^{m_1} - a_j\,z_2^{m_2}\right)}, \frac{z_2}{\prod_{j=1}^N \left(z_1^{m_1} - a_j\,z_2^{m_2}\right)^n}\right)$	$\begin{pmatrix} \frac{\lambda_1}{\lambda_2^{1/n}} & 0 \\ 0 & \frac{\lambda_1^{m_1 N}}{\lambda_2^{1/n}} \end{pmatrix}$

Table 3: The hyperresonant structures on hyperresonant Hopf surfaces. The quantities $a_1, a_2, \dots, a_N$ are any distinct nonzero complex constants. The holonomy generator in each case is actually $(g, 0)$ where g is the matrix given in the last column above. When a matrix contains an n-th root, like $\lambda_1^{1/n}$, the same value of the n-th root must be used in every entry in that matrix.

6.4 The hyperresonant structures on hyperresonant Hopf surfaces

Definition 6.1. A hyperresonant Hopf surface with hyperresonance $\lambda_1^{m_1} = \lambda_2^{m_2}$ may have additional $\mathcal{O}(n)$-structures, which we will call *hyperresonant* structures. It has such structures just when it satisfies the conditions given in table 3, as we will see in section 7.1.1 on page 26.

None of these are complete or essential structures, as the reader can easily check. The images in $\mathcal{O}(n)$ of the developing maps are complicated. The developing maps cover their images as finite unramified covering maps, with more than one sheet.

7 Classification on Hopf surfaces

Theorem 7.1. *Up to isomorphism, the $\mathcal{O}(n)$-structures on Hopf surfaces are precisely those given in table 4 on the next page, i.e. precisely the examples given in tables 2 on page 21 and 3.*

The proof of this theorem will occupy the remainder of this section. Note that every Hopf surface admits an $\mathcal{O}(n)$-structure for some value of n. Every

	structure		
surface	radial	eigenstructure	hyperresonant
generic	✓	✓	×
hyperresonant	✓	✓	✓
exceptional linear	✓	✓	×
exceptional nonlinear	×	✓	×

Table 4: The classification

structure	complete	essential
radial	×	×
eigenstructure	×	✓
hyperresonant	×	×

Table 5: Completeness and essentiality

linear Hopf surface admits an $\mathcal{O}(n)$-structure for all $n \geq 1$. Every nonlinear Hopf surface only admits $\mathcal{O}(n)$-structures for $n \geq m$ where m is the degree of the Hopf surface.

Roughly speaking, even among hyperresonant Hopf surfaces, hyperresonant structures are somewhat rare. To be precise: a hyperresonant Hopf surface admits a hyperresonant $\mathcal{O}(n)$-structure if and only if its hyperresonance (m_1, m_2) has $m_2 = n\, m_1$ for some integer n, and then either (1) only admits hyperresonant $\mathcal{O}(n)$-structures for that integer n or (2) if $m_1 = m_2$, admits hyperresonant $\mathcal{O}(n)$-structures for all $n \geq 1$.

7.1 Diagonal Hopf surfaces

Let's suppose that S_F is a Hopf surface and F is diagonal linear, say

$$F(z) = (\lambda_1 z_1, \lambda_2 z_2)\,.$$

Each $\mathcal{O}(n)$-structure on S_F has developing map a local biholomorphism $\operatorname{dev} : \mathbb{C}^2 \setminus 0 \to \mathcal{O}(n)$. There is a holonomy generator $\operatorname{hol} = (g, p) \in G$ so that $\operatorname{dev}(F(z)) = (g, p)\operatorname{dev}(z)$. In the coordinates (t_1, t_2) we constructed above on $\mathcal{O}(n)$, the developing map is a pair of meromorphic functions $(t_1, t_2) = (t_1(z_1, z_2), t_2(z_1, z_2))$. In particular, $t_1(\lambda_1 z_1, \lambda_2 z_2) = g\, t_1(z_1, z_2)$, where g acts here by linear fractional transformation. So t_1 is a meromorphic section of a $\mathbb{P}^1$-bundle over our Hopf surface. By proposition 3.3 on page 7, t_1 must have the form

$$t_1(z_1, z_2) = z_1^{k_1} z_2^{k_2} \frac{P_1(u)}{Q_1(u)} \quad \text{where } u = z_1^{m_1}/z_2^{m_2},$$

and either (1) P_1 and Q_1 are constants or (2) $\lambda_1^{m_1} = \lambda_2^{m_2}$ is a hyperresonance. At the expense of changing the values of k_1 and k_2, we can assume that $P_1(u)$

and $Q_1(u)$ each have no zeroes at $u = 0$, and that they have no zeroes in common. The map t_1 is the composition $\mathbb{C}^2 \setminus 0 \to \mathcal{O}(n) \to \mathbb{P}^1$, a composition of holomorphic submersions, so a holomorphic submersion. Therefore $k_1 = -1, 0$ or 1, and P_1 and Q_1 can't have multiple zeroes. The map t_1 as written is not defined at $z_2 = 0$, and we have to rewrite it in order to examine its behaviour near $z_2 = 0$. It is convenient to rewrite the map as

$$\begin{aligned} t_1(z_1, z_2) &= z_1^{\tilde{k}_1} z_2^{\tilde{k}_2} \frac{\tilde{P}_1(\tilde{u})}{\tilde{Q}_1(\tilde{u})} \quad \text{where} \\ \tilde{u} &= \frac{1}{u} = z_2^{m_2}/z_1^{m_1}, \\ \tilde{k}_1 &= k_1 + m_1 (\deg P_1 - \deg Q_1) \\ \tilde{k}_2 &= k_2 - m_2 (\deg P_1 - \deg Q_1) \end{aligned}$$

and the roots of $\tilde{P}_1$ and $\tilde{Q}_1$ are the reciprocals of those of P_1 and Q_1. Then $\tilde{k}_2$ must also be among $-1, 0$, or 1. These conditions together ensure that $t_1 : \mathbb{C}^2 \setminus 0 \to \mathbb{P}^1$ is a holomorphic submersion, and satisfies $t_1(F(z)) = g\, t_1(z)$. Moreover, they force g to have the form

$$g = \begin{pmatrix} c\lambda_1^{k_1} \lambda_2^{k_2} & 0 \\ 0 & c \end{pmatrix}$$

with $k_1 = -1, 0$ or 1 and $\tilde{k}_2 = -1, 0$ or 1 and $c \neq 0$.

7.1.1 Generic holonomy on diagonal Hopf surfaces

Consider a diagonal Hopf surface with $\mathcal{O}(n)$-structure. Assume that the holonomy generator (g, p) is generic, i.e. has the form $(g, 0)$, and the surface is diagonal.

Definition 7.2. A *semiadmissible map* for a diagonal Hopf surface S_F with

$$F = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and with hyperresonance $\lambda_1^{m_1} = \lambda_2^{m_2}$ is a map $t : \mathbb{C}^2 \setminus 0 \to \mathcal{O}(n)$ of the form

$$(t_1, t_2) = \left(z_1^{k_1} z_2^{k_2} \frac{P_1(u)}{Q_1(u)}, z_1^{\ell_1} z_2^{\ell_2} \frac{P_2(u)}{Q_1(u)^n} \right) \tag{1}$$

$$(s_1, s_2) = \left(z_1^{-k_1} z_2^{-k_2} \frac{Q_1(u)}{P_1(u)}, z_1^{\ell_1 - nk_1} z_2^{\ell_2 - nk_2} \frac{P_2(u)}{P_1(u)^n} \right), \tag{2}$$

so that

1. the expressions $P_1(u), Q_1(u), P_2(u)$ are polynomials, where $u = z_1^{m_1}/z_2^{m_2}$ and
2. none of these polynomials have any double roots, or roots at $u = 0$, and

3. no two of them have any common roots and

4. k_1 and ℓ_1 belong to the following list:

k_1	ℓ_1
-1	$-n$
0	0
0	1
1	0

   and

5. the numbers $\tilde{k}_2, \tilde{\ell}_2$ belong to this same list, where $\tilde{k}_2 = k_2 - m_2 (\deg P_1 - \deg Q_1)$ and $\tilde{\ell}_2 = \ell_2 - m_2 (\deg P_1 - n \deg Q_1)$.

(We will discuss semiadmissible maps in this section only.)

Lemma 7.3. *Suppose that $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a diagonal linear map in the Poincaré domain. A map $t : \mathbb{C}^2 \setminus 0 \to \mathcal{O}(n)$ is semiadmissible for the Hopf surface S_F if and only if it is the developing map of a branched $\mathcal{O}(n)$-structure on S_F. The holonomy of the branched $\mathcal{O}(n)$-structure is $(g, 0)$ where*

$$g = \begin{pmatrix} \lambda_1^{k_1+\ell_1} \lambda_2^{k_2+\ell_2} & 0 \\ 0 & \lambda_1^{\ell_1} \lambda_2^{\ell_2} \end{pmatrix}.$$

Proof. A semiadmissible map is equivariant under the action of F, so provides a branched $\mathcal{O}(n)$-structure.

Clearly $t_2(\lambda_1 z_1, \lambda_2 z_2) = \frac{1}{c^n} t_2(z_1, z_2)$. So t_2 is also a meromorphic section of a $\mathbb{P}^1$-bundle on the Hopf surface. Therefore $c^n = \lambda_1^{\ell_1} \lambda_2^{\ell_2}$ for some integers ℓ_1 and ℓ_2, and

$$t_2(z_1, z_2) = z_1^{\ell_1} z_2^{\ell_2} \frac{P_2(u)}{Q_2(u)} \quad \text{where } u = z_1^{m_1}/z_2^{m_2}.$$

Again, we can assume that P_2 and Q_2 have no roots at 0 and no roots in common. To have (t_1, t_2) a local biholomorphism, P_2 can't have any double roots, and $\ell_1, \ell_2 \leq 1$. One of the two mappings

$$\begin{aligned} (t_1, t_2) &= \left(z_1^{k_1} z_2^{k_2} \frac{P_1(u)}{Q_1(u)}, z_1^{\ell_1} z_2^{\ell_2} \frac{P_2(u)}{Q_2(u)} \right) \\ (s_1, s_2) &= \left(z_1^{-k_1} z_2^{-k_2} \frac{Q_1(u)}{P_1(u)}, z_1^{\ell_1 - nk_1} z_2^{\ell_2 - nk_2} \frac{P_2(u) Q_1(u)^n}{Q_2(u) P_1(u)^n} \right) \end{aligned}$$

must be defined at each point $(z_1, z_2) \in \mathbb{C}^2 \setminus 0$. It is not possible for t_1 and t_2 to have a common polynomial factor, because where this polynomial vanishes, the developing map will not be a local biholomorphism. When we look at the line $z_1 = 0$, we see that this constrains us to the table of (k_1, ℓ_1) values above, and exactly the same is true of $\left(\tilde{k}_2, \tilde{\ell}_2\right)$ by the same reasoning. Clearly P_2 cannot have any double zeroes, or else (t_1, t_2) won't be a local biholomorphism. The

functions t_1, t_2 cannot be defined where $Q_1 = 0$ or where $Q_2 = 0$, so (s_1, s_2) must be defined. Therefore each zero of Q_2 must cancel a zero of Q_1^n. But then the leftover zeroes of Q_1^n cannot be double zeroes, so each zero of Q_2 must occur with multiplicity precisely $n-1$ or n. We can write $Q_2 = RQ_1^{n-1}$, with R dividing Q_1 and having no double roots. So we can write $Q_1 = RS$, with R and S having no common roots. It then follows that S is a factor of s_1 and s_2, so s_1 and s_2 have linearly dependent differentials at points where $S = 0$. These are points at which both s_1 and s_2 are holomorphic. Therefore S must be constant. Absorbing the constant, we have $Q_2 = Q_1^n$. □

Calculation on any semiadmissible map yields

$$\begin{aligned}
\frac{\partial t_1}{\partial z_1} &= \frac{t_1}{z_1}\left(k_1 + m_1 u \left(\frac{P_1'}{P_1} - \frac{Q_1'}{Q_1}\right)\right), \\
\frac{\partial t_1}{\partial z_2} &= \frac{t_1}{z_2}\left(k_2 - m_2 u \left(\frac{P_1'}{P_1} - \frac{Q_1'}{Q_1}\right)\right), \\
\frac{\partial t_2}{\partial z_1} &= \frac{t_2}{z_1}\left(\ell_1 + m_1 u \left(\frac{P_2'}{P_2} - n\frac{Q_1'}{Q_1}\right)\right), \\
\frac{\partial t_2}{\partial z_2} &= \frac{t_2}{z_2}\left(\ell_2 - m_2 u \left(\frac{P_2'}{P_2} - n\frac{Q_1'}{Q_1}\right)\right).
\end{aligned}$$

and so

$$\det t' = z_1^{k_1+\ell_1-1} z_2^{k_2+l_2-1} \frac{P_1(u)P_2(u)}{Q_1(u)^{n+1}} R(u)$$

where

$$R(u) = A\,\frac{u\,P_1'(u)}{P_1(u)} - B\,\frac{u\,P_2'(u)}{P_2(u)} + C\,\frac{u\,Q_1'(u)}{Q_1(u)} + D,$$

and

$$\begin{aligned}
A &= m_1\ell_2 + \ell_1 m_2 \\
B &= m_1 k_2 + k_1 m_2 \\
C &= n\left(m_1 k_2 + k_1 m_2\right) - \left(m_1 \ell_2 + \ell_1 m_2\right) \\
D &= k_1 \ell_2 - \ell_1 k_2.
\end{aligned}$$

We can see that semiadmissible maps are local biholomorphisms near $z_1 = 0$ and near $z_2 = 0$. They determine branched $\mathcal{O}(n)$-structures. However, in order that the branched structure of a semiadmissible map be an unbranched structure, $\det t'$ must have no zeroes except at points where (t_1, t_2) are not defined. Recall that if

$$P_1(u) = c \prod_j (u - a_j)$$

then

$$\frac{uP_1'(u)}{P_1(u)} = u \sum_j \frac{1}{u - a_j}.$$

The function $R(u)$ has simple poles at the zeroes of $P_1(u)$ if $A \neq 0$, at the zeroes of $P_2(u)$ if $B \neq 0$, and at the zeroes of $Q_1(u)$ if $C \neq 0$. If $R(u)$ has any zeroes at finite values of u, then in order to keep $\det t' \neq 0$, we would need to have those zeroes occur somewhere where they can cancel out with poles from $z_1^{k_1+\ell_1-1} z_2^{k_2+l_2-1} \frac{P_1(u)P_2(u)}{Q_1(u)^{n+1}}$. So the finite zeroes of $R(u)$ only occur at zeroes of $Q_1(u)$.

If $C \neq 0$, then $R(u)$ has poles at all of the zeroes of $Q_1(u)$, so no cancellations take place. But then $R(u)$ must have at least $\deg Q_1$ zeroes (counting with multiplicity), so these must lie at infinity. If $Q_1(u)$ is constant, then no cancellations can take place, so $R(u)$ can't have any zeroes at finite values of u, so again all zeroes of $R(u)$ are at infinity. Therefore $Q_1(u)$ is constant or $C = 0$ or all zeroes of $R(u)$ are at infinity.

Definition 7.4. A semiadmissible map is *admissible* if (in the above notation from this section)

1. either $A = 0$ or $P_1(u)$ is constant and
2. either $B = 0$ or $P_2(u)$ is constant and
3. either $C = 0$ or $Q_1(u)$ is constant and
4. $R(u) = D$ is constant, not zero and
5. $k_1 \neq 0$ or $\ell_1 \neq 0$.

(We will discuss admissible maps in this section only.)

Lemma 7.5. *A semiadmissible map which is local biholomorphism at every point of* $\mathbb{C}^2 \setminus 0$ *(i.e. not branched) is admissible.*

Proof. Suppose that $A \neq 0$. Let's pick one of the zeroes of $P_1(u)$, say a_1, and write $R(u)$ as

$$R(u) = \frac{Au}{u - a_1} + f(u).$$

Now solve $R(u) = 0$ for a_1:

$$a_1 = u\left(1 + \frac{A}{f}\right).$$

Imagine varying the choice of P_1, by varying a_1 and leaving the other linear factors of P_1 intact. We thereby vary the choice of t and so of R. We see that in order to force $R(u) = 0$ at a given value of u, we only have to set a_1 as above. For generic choice of u, there is therefore a unique choice of a_1 which will ensure $R(u) = 0$. So if we $P_1(u)$ is not constant, we can slightly alter $P_1(u)$ to ensure that $R(u)$ has all its zeroes at finite locations away from the zeroes of $Q_1(u)$. Therefore generic choice of $P_1(u)$ will lead to a branched $\mathcal{O}(n)$-structure, which has a nontrivial branch locus. The limit of the branch locus is still a compact curve in the Hopf surface, since the space of curves is compact. Therefore if

$A \neq 0$ and $P_1(u)$ is not constant, then the branched structure has nonempty branch locus. The same proof works for $P_2(u)$.

By the same argument, if $Q_1(u)$ is not constant, and $C \neq 0$, then we can perturb to a branched structure, which has finite roots for $R(u)$. Therefore this perturbed structure must have nonempty branch locus, and so our original branched structure had nonempty branch locus. □

Lemma 7.6. *A semiadmissible map is admissible if and only if its branch locus is empty, i.e. it is the developing map of an $\mathcal{O}(n)$-structure.*

Proof. If we have an admissible map then our branched structure has no branch locus in the region in which the (t_1, t_2) functions in equation 1 are defined. Therefore we only need to then check the branch locus in all four coordinate charts: the (t_1, t_2) and (s_1, s_2) charts on $\mathcal{O}(n)$, and the expressions in z_1, z_2 with and without ˜ symbols on them, the charts on the Hopf surface. It is easy to check that when we change to the ˜ symbol coordinates, the corresponding quantities, in the obvious notation, are

$$\begin{aligned}
\tilde{u} &= \frac{1}{u} = z_2^{m_2}/z_1^{m_1}, \\
\tilde{k}_1 &= k_1 + m_1 \left(\deg P_1 - \deg Q_1\right), \\
\tilde{k}_2 &= k_2 - m_2 \left(\deg P_1 - \deg Q_1\right), \\
\tilde{m}_1 &= -m_1, \\
\tilde{m}_2 &= -m_2, \\
\tilde{\ell}_1 &= \ell_1 + m_1 \left(\deg P_2 - n \deg Q_1\right), \\
\tilde{\ell}_2 &= \ell_2 - m_2 \left(\deg P_2 - n \deg Q_1\right), \\
\tilde{A} &= -A, \\
\tilde{B} &= -B, \\
\tilde{C} &= -C, \\
\tilde{D} &= D + A \left(\deg P_1 - \deg Q_1\right) - B \left(\deg P_2 - n \deg Q_1\right).
\end{aligned}$$

We can easily see that $\tilde{P}_1(\tilde{u})$ is constant just when $P_1(u)$ is constant and $\tilde{A} = 0$ just when $A = 0$, etc. Therefore admissibility is unchanged by such a coordinate transformation.

Let's write out our map in (s_1, s_2) coordinates, say

$$\begin{aligned}
(s_1, s_2) &= \left(z_1^{\hat{k}_1} z_2^{\hat{k}_2} \frac{\hat{P}_1(u)}{\hat{Q}_1(u)}, z_1^{\hat{\ell}_1} z_2^{\hat{\ell}_2} \frac{\hat{P}_2(u)}{\hat{Q}_1(u)^n} \right) \\
&= \left(z_1^{-k_1} z_2^{-k_2} \frac{Q_1(u)}{P_1(u)}, z_1^{\ell_1 - nk_1} z_2^{\ell_2 - nk_2} \frac{P_2(u)}{P_1(u)^n} \right).
\end{aligned}$$

Then we find the dictionary

$$\begin{aligned}
\hat{k}_1 &= -k_1, \\
\hat{k}_2 &= -k_2, \\
\hat{m}_1 &= m_1, \\
\hat{m}_2 &= m_2, \\
\hat{l}_1 &= \ell_1 - n\,k_1, \\
\hat{l}_2 &= \ell_2 - n\,k_2, \\
\hat{A} &= A - n\,B, \\
\hat{B} &= -B, \\
\hat{C} &= -A, \\
\hat{D} &= -D.
\end{aligned}$$

Again, the admissibility of a semiadmissible map is unchanged by this coordinate transformation. $\square$

By semiadmissibility, we need $D \neq 0$, so $(k_1, \ell_1) \neq (0,0)$. By admissibility, we will also need $\tilde{D} \neq 0$ so $\left(\tilde{k}_2, \tilde{\ell}_2\right) \neq (0,0)$. We now have a tedious computation: for each of the 3 possible values of (k_1, ℓ_1) and the 3 possible values of $\left(\tilde{k}_2, \tilde{\ell}_2\right)$ from table 4 on page 27 (except for $(0,0)$), we calculate $A, B, C, D, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$. In order that P_1, Q_1 and P_2 not be all constant, we will need one of $\deg P_1, \deg Q_1, \deg P_2$ to be nonzero. If P_1 is not constant, then $A = 0$, etc. To keep the branched $\mathcal{O}(n)$-structure from branching, we will need $D \neq 0, \tilde{D} \neq 0$, and $\hat{D} \neq 0$. The full story of manipulating these inequalities is totally elementary, so we will only explain fully one of the nine cases, and then leave all others to the reader, to avoid many pages of elementary arguments with inequalities.

Consider the case of an admissible map with $k_1 = 0, \ell_1 = 1, \tilde{k}_2 = -1, \tilde{\ell}_2 = -n$. We find

$$\begin{aligned}
A &= m_1 n - m_2 - m_1 m_2 \left(\deg P_2 - n \deg Q_1\right), \\
B &= m_1 \left(-1 + m_2 \left(\deg P_1 - \deg Q_1\right)\right), \\
C &= m_2 \left(m_1 n \deg P_1 - m_1 \deg P_2 - 1\right), \\
D &= -1 + m_2 \left(\deg P_1 - \deg Q_1\right).
\end{aligned}$$

In particular, $B = m_1 D$ and $m_1 \neq 0$ because the hyperresonance has $m_1, m_2 \geq 1$. Moreover $D \neq 0$, since the structure is not branched. Therefore $B \neq 0$. By admissibility, P_2 is a constant. We compute that $\hat{D} = -1 - m_1 \deg P_2 + m_1 n \deg P_1$. Therefore $C = m_2 \hat{D}$. We can therefore say that $C \neq 0$ and so Q_1 is constant by admissibility. Assume that not all of P_1, P_2, Q_1 are constant. Clearly now $A = m_1\,n - m_2$, and so $m_2 = m_1 n$. Plugging this in gives $0 \neq D = -1 + m_1 n \deg P_1$, so $m_1 n \deg P_1 \neq 1$, and so $m_1 n \deg P_1 > 1$ since

k_1	ℓ_1	$\tilde{k}_2$	$\tilde{l}_2$	$\deg P_1$	$\deg Q_1$	$\deg P_2$	and...
0	1	-1	$-n$	≥ 1	0	0	$m_2 = n\, m_1$, $m_1 > 1$ or $n > 1$ or $\deg P_1 > 1$
0	1	0	1	$\frac{m_1+m_2}{m_1 m_2 n}$	$\frac{m_1+m_2}{m_1 m_2 n}$	0	$\deg P_1 \neq \deg Q_1$, impossible
0	1	1	0	0	≥ 1	0	$m_2 = n\, m_1$, $m_1 > 1$ or $n > 1$ or $\deg Q_1 > 1$
1	0	-1	$-n$	0	0	≥ 1	$m_1 = m_2$, $m_1 \deg P_2 \neq n$
1	0	0	1	0	≥ 1	0	$m_1 = n\, m_2$, $m_2 > 1$ or $n > 1$ or $\deg Q_1 > 1$
1	0	1	0	0	$\frac{m_1+m_2}{m_1 m_2}$	$\frac{n(m_1+m_2)}{m_1 m_2}$	$\deg P_2 \neq n \deg Q_1$, impossible

Table 6: The 6 cases of developing maps with at least one of P_1, Q_1, P_2 not constant

k_1	ℓ_1	$\tilde{k}_2$	$\tilde{l}_2$	t_1	t_2
0	1	-1	$-n$	$\frac{1}{z_2}$	$\frac{z_1}{z_2^n}$
0	1	1	0	z_2	z_1
1	0	-1	$-n$	$\frac{z_1}{z_2}$	$\frac{1}{z_2^n}$
1	0	0	1	z_1	z_2

Table 7: The possible values of the integers k_1, ℓ_1, k_2, ℓ_2 and associated developing maps if P_1, Q_1 and P_2 are constant

$m_1, n, \deg P_1 \geq 1$. Therefore $m_1 > 1$ or $n > 1$ or $\deg P_1 > 1$. All eight other cases follow essentially the same reasoning.

Tedious computation of all nine cases yields the conditions of table 6 in order that the structure is unbranched and $P_1(u), P_2(u)$ and $Q_1(u)$ are not all forced to be constant. The impossible cases come from inconsistency of the degrees of the polynomials P_1, Q_1, and P_2.

On the other hand, if P_1, Q_1 and P_2 are all assumed to be constants, and the structure is unbranched (i.e. $D \neq 0$) then we can arrange that $k_1 \geq 0$ by replacing (t_1, t_2) coordinates by (s_1, s_2) coordinates, and then we see that the possible values of k_1, ℓ_1, k_2 and ℓ_2 are given in table 7.

Note that with P_1, Q_1 and P_2 constants, we can rescale z_1 and z_2 independently, since these rescalings commute with our linear map F, and thereby absorb constants as needed. After such absorptions, we find the developing maps in table 7. It is easy to see that the first line of this table is isomorphic to the second via the isomorphism $(g, p) = (g, 0)$ with

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly the second and fourth lines of table 7 are eigenstructures, while the third is a radial structure.

Proposition 7.7. *On any generic Hopf surface, the $\mathcal{O}(n)$-structures with generic holonomy, up to isomorphism, are the radial structures and eigenstructures.*

Proof. Just plug in the values in table 7 on the preceding page into the general expression of a semiadmissible map, with the added information that all of the polynomials in u must be constants, to find that up to isomorphism:

developing map	holonomy generator
$\left(\frac{1}{z_2}, \frac{z_1}{z_2^n}\right)$	$\begin{pmatrix} \frac{1}{\lambda_1^{1/n}} & 0 \\ 0 & \frac{\lambda_2}{\lambda_1^{1/n}} \end{pmatrix}$
(z_2, z_1)	$\begin{pmatrix} \frac{\lambda_2}{\lambda_1^{1/n}} & 0 \\ 0 & \frac{1}{\lambda_1^{1/n}} \end{pmatrix}$
$\left(\frac{z_1}{z_2}, \frac{1}{z_2^n}\right)$	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
(z_1, z_2)	$\begin{pmatrix} \frac{\lambda_1}{\lambda_2^{1/n}} & 0 \\ 0 & \frac{1}{\lambda_2^{1/n}} \end{pmatrix}$

□

Proposition 7.8. *On any hyperresonant Hopf surface, with hyperresonance $\lambda_1^{m_1} = \lambda_2^{m_2}$, the $\mathcal{O}(n)$-structures with generic holonomy, up to isomorphism, are precisely*

1. *the radial structures,*
2. *the eigenstructures, and*
3. *the hyperresonant structures.*

Proof. Just plug in the values from table 6 on the previous page and you find table 3 on page 24. □

7.1.2 Nongeneric holonomy on diagonal Hopf surfaces

Lemma 7.9. *Suppose that (g, p) is the holonomy of an $\mathcal{O}(n)$-structure on a Hopf surface. Then g has infinite order, i.e. $g^N \neq I$ for any integer $N \neq 0$.*

Proof. Suppose that g has finite order, say $g^N = I$, $N \geq 1$. By lemma 5.9 on page 18, we can assume that (g, p) is in normal form. In particular we can assume that g is diagonal. Suppose that (g, p) is the holonomy of an $\mathcal{O}(n)$-structure on a Hopf surface S_F. Let F^N be the N-fold composition $F \circ F \circ \ldots F$. Then $\left(g^N, Np\right)$ is the holonomy of the pullback $\mathcal{O}(n)$-structure on the Hopf surface S_{F^N} via the obvious covering map $S_{F^N} \to S_F$. Therefore, by possibly replacing S_F with S_{F^N}, we can assume that $g = I$.

Suppose that the developing map of the $\mathcal{O}(n)$-structure, in affine coordinates, is $(t_1, t_2) : \mathbb{C}^2 \setminus 0 \to \mathcal{O}(n)$. Because $g = I$, t_1 must be F-invariant,

i.e. $t_1 : S_F \to \mathbb{C}$ is a nonconstant rational function. Therefore F must be a hyperresonant map, say

$$F(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2),$$

for some complex numbers λ_1, λ_2 with $0 < |\lambda_1| \le |\lambda_2| < 1$, with hyperresonance $\lambda_1^{m_1} = \lambda_2^{m_2}$. Moreover, $t_1 = P(u)/Q(u)$ for some polynomials P and Q, where $u = z_1^{m_1}/z_2^{m_2}$. The function t_2 must then satisfy

$$t_2(\lambda_1 z_1, \lambda_2 z_2) = t_2(z_1, z_2) + p(t_1(z_1, z_2), 1).$$

The developing map is a local biholomorphism, so $t_1 : S_F \to \mathbb{P}^1$ is a submersion to $\mathbb{P}^1$. We can factor t_1 into $t_1 = (P/Q) \circ u$, and so $P/Q : \mathbb{P}^1 \to \mathbb{P}^1$ must be a local biholomorphism, and so a linear fractional transformation. After replacing our developing map and holonomy by $\left((g, 0) \operatorname{dev}, (g, 0) \operatorname{hol}(g, 0)^{-1}\right)$, using an element $(g, 0) \in G$, we can arrange that $P(u)/Q(u) = u$, i.e. $t_1(z_1, z_2) = u = z_1^{m_1}/z_2^{m_2}$. Clearly $u : \mathbb{C}^2 \setminus 0 \to \mathbb{P}^1$ must be a submersion, and so $m_1 = m_2 = 1$, i.e. the hyperresonance is $\lambda_1 = \lambda_2$, and we can write $F(z_1, z_2) = (\lambda z_1, \lambda z_2)$ for some $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$.

If we pick any point where (t_1, t_2) are not defined as complex valued functions, then at that point (s_1, s_2) must be defined. So

$$(s_1, s_2) = \left(\frac{z_2}{z_1}, \frac{z_2^n t_2}{z_1^n}\right)$$

must be defined. In particular, $z_2^n s_2$ must be defined at such a point. So the function $T = z_2^n s_2$ is defined and holomorphic at every point of $\mathbb{C}^2 \setminus 0$ and so at every point of $\mathbb{C}^2$. Since we know how t_2 behaves under holonomy action, we find

$$T(\lambda z_1, \lambda z_2) = \lambda^n T(z_1, z_2) + \lambda^n p(z_1, z_2).$$

Expanding T into a power series, we find that $p = 0$.

So now $(g, p) = (I, 0)$, and t_1 and t_2 are both F-invariant meromorphic functions on $\mathbb{C}^2 \setminus 0$, i.e. meromorphic functions on S_F, i.e. rational functions of $u = z_1^{m_1}/z_2^{m_2}$, so $dt_1 \wedge dt_2 = 0$. But then $\operatorname{dev} = (t_1, t_2)$ is not a local biholomorphism. $\square$

Lemma 7.10. *The holonomy generator (g, p) of any $\mathcal{O}(n)$-structure on any diagonalizable Hopf surface, up to conjugation, has $p = 0$ or p a monic monomial.*

Proof. By lemma 5.9 on page 18, either $p = 0$ or p is monomial or g has finite order. Finite order g is impossible by lemma 7.9 on the previous page. $\square$

Lemma 7.11. *The holonomy of any $\mathcal{O}(n)$-structure on any diagonalizable Hopf surface is generic.*

Proof. We can assume that the map F determining our Hopf surface is linear, diagonal,

$$F(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2),$$

and hyperresonant, with hyperresonance (m_1, m_2). Suppose that $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a diagonal linear map in the Poincaré domain, $F(z) = (\lambda_1 z_1, \lambda_2 z_2)$. Suppose that $\text{dev} : \mathbb{C}^2 \setminus 0 \to \mathcal{O}(n)$ is the developing map of an $\mathcal{O}(n)$-structure on the associated Hopf surface S_F, with holonomy generator $\text{hol} = (g, p)$. Then in affine coordinates, this developing map has the form (t_1, t_2), where t_1 must be a section of a flat projective line bundle associated to (F, g). By proposition 3.3 on page 7,

$$g = \begin{pmatrix} \lambda_1^{k_1} \lambda_2^{k_2} c & 0 \\ 0 & c \end{pmatrix}$$

for some nonzero complex number c, and

$$t_1 = z_1^{k_1} z_2^{k_2} \frac{P_1(u)}{Q_1(u)}$$

for $u = z_1^{m_1}/z_2^{m_2}$. (If g is not hyperresonant, we take $P_1(u)$ and $Q_1(u)$ to be constants.) We can assume that neither of P_1 and Q_1 have any double roots, or roots at $u = 0$, and that neither of them have any common roots, and that $k_1 = -1, 0, 1$ and $k_2 = -1, 0, 1$ as before since $t_1 : \mathbb{C}^2 \backslash 0 \to \mathbb{P}^1$ is a submersion.

By lemma 7.10 on the previous page, if the holonomy (g, p) is not generic, then we arrange that p is a monic monomial, say

$$g = \begin{pmatrix} \lambda_1^{k_1} \lambda_2^{k_2} c & 0 \\ 0 & c \end{pmatrix}$$

for some nonzero complex number c, and

$$p(Z_1, Z_2) = Z_1^k Z_2^{n-k},$$

and

$$t_1 = z_1^{k_1} z_2^{k_2} \frac{P_1(u)}{Q_1(u)}$$

for $u = z_1^{m_1}/z_2^{m_2}$. We can assume that neither of these polynomials have any double roots, or roots at $u = 0$, and that neither of them have any common roots, and that $k_1 = -1, 0, 1$ and $k_2 = -1, 0, 1$. In order that p be resonant, we will need

$$\left(\lambda_1^{k_1} \lambda_2^{k_2} c\right)^k c^{n-k} = 1,$$

i.e.

$$c^n = \lambda_1^{-kk_1} \lambda_2^{-kk_2}.$$

Next consider t_2. At this stage, we can see that

$$t_2(\lambda_1 z_1, \lambda_2 z_2) = \lambda_1^{kk_1} \lambda_2^{kk_2} t_2(z_1, z_2) + z_1^{kk_1} z_2^{kk_2} \frac{P_1(u)^k}{Q_1(u)^k}.$$

Let

$$f(z_1, z_2) = \frac{t_2(z_1, z_2)}{z_1^{kk_1} z_2^{kk_2} \frac{P_1(u)^k}{Q_1(u)^k}}.$$

Then compute out

$$f(F(z)) = f(z) + \frac{1}{\lambda_1^{kk_1}\lambda_2^{kk_2}},$$

so that f is a meromorphic section of a flat projective line bundle,

$$\left(\mathbb{C}^2 \setminus 0\right)_{(F,g')} \mathbb{P}^1,$$

where

$$g' = \begin{bmatrix} \lambda_1^{kk_1}\lambda_2^{kk_2} & 1 \\ 0 & \lambda_1^{kk_1}\lambda_2^{kk_2} \end{bmatrix}.$$

By proposition 3.3 on page 7, the only meromorphic section of this line bundle is $f = \infty$, a contradiction. □

Summing up:

Proposition 7.12. *The only $\mathcal{O}(n)$-structures on diagonalizable Hopf surfaces are*

1. *the radial structures,*
2. *the eigenstructures and*
3. *the hyperresonant structures in table 3 on page 24.*

7.2 Exceptional Hopf surfaces

7.2.1 Diagonalizable holonomy on exceptional Hopf surfaces

Proposition 7.13. *Up to isomorphism, the only $\mathcal{O}(n)$-structure on an exceptional Hopf surface which has diagonalizable holonomy is the eigenstructure.*

Proof. Suppose that $(t_1, t_2) : \mathbb{C}^2 \backslash 0 \to \mathcal{O}(n)$ is the developing map of an $\mathcal{O}(n)$-structure on an exceptional Hopf surface S_F, where $F(z_1, z_2) = (\lambda z_1, \lambda^m z_2 + z_1^m)$. Suppose that the holonomy is (g, p), and that g is diagonalizable, say

$$g = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

By proposition 3.3 on page 7, up to isomorphism we must have

$$t_1 = c\, z_1^k,$$

some integer k, and $a_1/a_2 = \lambda^k$. But then either t_1 is branched, along $z_1 = 0$, if $k > 1$, or else

$$s_1 = \frac{1}{t_1} = \frac{1}{c\, z_1^k}$$

is branched if $k < 1$. Up to isomorphism, we can therefore ensure that $k = 1$, and that

$$t_1 = z_1,$$

and $a_1/a_2 = \lambda$ and

$$t_2(F(z)) = \frac{t_2(z)}{a_2^n} + p(\lambda z_1, 1).$$

By the usual trick of writing t_2 in terms of Weierstrass polynomials,

$$t_2(z) = h(z)\frac{W_1(z)}{W_2(z)}$$

with $W_1(z)$ and $W_2(z)$ polynomial in the variable z_2, we see that $W_2(z)$ must transform under composition with F by scaling, so

$$W_2(z) = c\, z_1^k$$

for some constant $c \neq 0$ and integer $k \geq 0$. We can write

$$t_2(z) = \frac{T(z)}{z_1^k}$$

for some holomorphic function $T(z)$ defined near the origin. Calculate that

$$T(F(z)) = \frac{\lambda^k}{a_2^n} T(z) + \lambda^k z_1^k p(\lambda z_1, 1).$$

Differentiate both sides with respect to z_2 to find

$$\frac{\partial T}{\partial z_2}(F(z)) = \frac{\lambda^{k-m}}{a_2^n}\frac{\partial T}{\partial z_2}(z).$$

so that $\frac{\partial T}{\partial z_2}$ is a section of a line bundle over an exceptional Hopf surface, so

$$\frac{\partial T}{\partial z_2} = c\, z_1^\ell$$

for some $\ell \geq 0$ and constant c and

$$a_2^n = \lambda^{k-\ell-m}.$$

So

$$T = c\, z_1^\ell z_2 + T_1(z_1),$$

for some holomorphic function $T_1(z_1)$, which is then forced to satisfy

$$T_1(\lambda z_1) = \lambda^{\ell+m} T_1(z_1) - c\lambda^\ell z_1^{\ell+m} + \lambda^k z_1^k p(\lambda z_1, 1).$$

Expand out p as

$$p(z_1, 1) = \sum_{j=0}^{n} C_j z_1^j,$$

and

$$T_1(z_1) = \sum_{j=0}^{\infty} b_j z_1^j,$$

(with the understanding that $C_j = 0$ when $j < 0$ or $j > n$ and that $b_j = 0$ when $j < 0$) to see that

$$\left(\lambda^j - \lambda^{\ell+m}\right) b_j = C_{j-k}\lambda^k - c\,\lambda^\ell \delta_{j=\ell+m}.$$

If we plug in $j = \ell + m$, we find

$$c = C_{\ell+m-k}\lambda^{k-\ell},$$

and $b_{\ell+m}$ is arbitrary. For all other values of $j \neq \ell + m$,

$$b_j = \frac{C_{j-k}}{\lambda^{j-k} - \lambda^{\ell+m-k}}.$$

We now see that $a_1 = \lambda a_2$ and that $a_2^n = \lambda^{k-\ell-m}$, giving the eigenvalues of g. Therefore (g, p) is generic as long as either $k > \ell + m$ (g contracting) or $n + k < \ell + m$ (g expanding). So we can assume that $p = 0$ or else that $k \leq \ell + m \leq n + k$. If $a_1^i a_2^{n-i} = 1$, then plugging in a_1 and then a_2^n, we find that $i = \ell + m - k$, and $0 \leq i \leq n$. By lemma 5.9 on page 18 there is only this one possible value of i giving a coefficient C_i of p which we can't assume is 0 without loss of generality. Therefore we can arrange

$$p\left(z_1, 1\right) = C_{\ell+m-k} z_1^{\ell+m-k}.$$

and

$$t_2 = C_{\ell+m-k}\lambda^{k-\ell}\, z_1^{\ell-k} z_2 + b\, z_1^{\ell+m-k}.$$

for some complex constant $b = b_{\ell+m}$.

To see if this structure is branched, compute

$$dt_1 \wedge dt_2 = C_{\ell+m-k}\lambda^{k-\ell} z_1^{\ell-k}\, dz_1 \wedge dz_2.$$

So the structure is unbranched except possibly at $z_1 = 0$. Since $s_1 = \frac{1}{z_1}$, s_1 is not defined at $z_1 = 0$, and therefore we cannot use the coordinates s_1, s_2 to fix up the branch locus at $z_1 = 0$. Therefore t_1 and t_2 must be defined at $z_1 = 0$, so that $k \leq \ell$. Moreover $dt_1 \wedge dt_2$ can't vanish at $z_1 = 0$, so $k = \ell$, yielding

$$t_2 = C_m z_2 + b\, z_1^m,$$

and

$$p\left(z_1, 1\right) = C_m z_1^m.$$

We can conjugate by a suitable isomorphism to arrange that $C_m = 1$ and that $b = 0$, so that our $\mathcal{O}\left(n\right)$-structure is the eigenstructure on the exceptional Hopf surface. □

7.2.2 Nondiagonalizable holonomy on exceptional Hopf surfaces

Proposition 7.14. *The only exceptional Hopf surfaces which admit $\mathcal{O}(n)$-structures with holonomy (g, p) with g nondiagonalizable are the linear nondiagonalizable Hopf surfaces. Up to isomorphism, the only such $\mathcal{O}(n)$-structures they admit are the radial ones.*

Proof. Suppose that $(t_1, t_2) : \mathbb{C}^2 \backslash 0 \to \mathcal{O}(n)$ is the developing map of an $\mathcal{O}(n)$-structure on an exceptional Hopf surface S_F, where $F(z_1, z_2) = (\lambda z_1, \lambda^m z_2 + z_1^m)$. Suppose that the holonomy is (g, p), and that g is not diagonalizable, say

$$g = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

By lemma 5.3 on page 16, we can assume either (1) $(g, p) = (g, 0)$ or else (2)

$$(g, p) = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, Z_1^n \right).$$

In either case, t_1 is a meromorphic section of the obvious flat projective line bundle. Let's consider case (2). By proposition 3.3 on page 7,

$$t_1 = z_2 \left(\frac{\lambda}{z_1} \right)^m + c$$

for some constant c. In particular, $t_1 = \infty$ at $z_1 = 0$. Therefore at $z_1 = 0$, s_1 and s_2 must be holomorphic. Clearly

$$s_1 = \frac{z_1^m}{\lambda^m z_2 + c\, z_1^m}.$$

Therefore s_1 branches along $z_1 = 0$ unless $m = 1$. We can check that

$$s_2(F(z_1, z_2)) = \frac{s_2(z_1, z_2) + (1 + s_1(z_1, z_2))^n}{(1 + s_1(z_1, z_2))^n}.$$

In particular, along the line $z_1 = 0$, we find that $s_1 = 0$ and so

$$s_2(0, \lambda z_2) = s_2(0, z_2) + 1.$$

Since s_2 is holomorphic on the entire line $z_1 = 0$, except perhaps at $z_2 = 0$, we can compute a Laurent series expansion for s_2 and see that the constant term is inconsistent.

Therefore we can assume that we are in case (1): $p = 0$. By proposition 3.3 on page 7, up to isomorphism we must have

$$t_1 = \frac{z_2}{a} \left(\frac{\lambda}{z_1} \right)^m.$$

But then

$$s_1 = \frac{1}{t_1} = \frac{a}{z_2} \left(\frac{z_1}{\lambda} \right)^m$$

is branched unless $m = 1$. So now let's assume that $m = 1$. Our Hopf surface is linear but not diagonalizable, and

$$t_1 = \frac{\lambda z_2}{a z_1}.$$

and

$$t_2\left(F(z)\right) = \frac{t_2\left(z_1, z_2\right)}{a^n}.$$

By proposition 3.3 on page 7, this ensures that $t_2 = c\, z_1^k$ for some integer k, and that

$$\frac{1}{a^n} = \lambda^k.$$

On the line $z_1 = 0$, t_1 is infinite, so s_1 and s_2 must be finite. Similarly, on $z_2 = 0$, s_1 is infinite, so t_1 and t_2 must be finite, and have linearly independent differentials. Note that

$$s_1 = \frac{1}{t_1} = \frac{a z_1}{\lambda z_2}$$

and

$$s_2 = \frac{t_2}{t_1^n} = \left(\frac{a z_1}{\lambda z_2}\right)^n c z_1^k.$$

Along $z_1 = 0$, s_2 must be finite and $ds_1 \wedge ds_2 \neq 0$. In particular t_2 has a pole of order no more than n along $z_1 = 0$. Compute

$$dt_1 \wedge dt_2 = -\frac{c\lambda k}{a} z_1^{k-2} dz_1 \wedge dz_2.$$

The only possible zero of this holomorphic 2-form occurs along the line $z_1 = 0$, but t_1 and t_2 are not defined there, so we turn to s_1 and s_2 to see what happens near $z_1 = 0$. Compute

$$ds_1 \wedge ds_2 = ck \frac{a^{n+1} z_1^{k+n}}{\lambda^{n+1} z_2^{n+2}} dz_1 \wedge dz_2.$$

To get this to give a finite nonzero value along $z_1 = 0$, we need $k = -n$. Finally, composing with $(\lambda_0\, I, 0)$ where $\lambda_0^n = c$ gives a developing map which is identical to the developing map of the radial structure, and gives the same holonomy. □

Summing up:

Corollary 7.15. *Up to isomorphism, the only $\mathcal{O}(n)$-structures on any exceptional Hopf surface are*

1. *the eigenstructure and*
2. *on a linear exceptional Hopf surface, the radial structure.*

This completes the proof of theorem 7.1 on page 24.

8 Locally homogeneous geometric structures inducing these structures

Lemma 8.1. *The Zariski closure of the subgroup of* $\mathrm{GL}(2, \mathbb{C})$ *generated by a matrix*

$$g = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

with neither a_1 *nor* a_2 *on the unit circle is*

$$\begin{array}{ll} \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \middle| \, Z_1^{n_1} = Z_2^{n_2} \right\} & \text{if } g \text{ has hyperresonance } a_1^{n_1} = a_2^{n_2} \\ \text{the diagonal matrices} & \text{if } g \text{ is not hyperresonant.} \end{array}$$

Proof. Suppose that $p(Z_1, Z_2)$ is a complex polynomial vanishing on all of the points $(Z_1, Z_2) = \left(a_1^k, a_2^k\right)$ for all integers k. From among all monomials $Z_1^{j_1} Z_2^{j_2}$ which occur in p with nonzero coefficient, pick one for which $a_1^{j_1} a_2^{j_2}$ is largest in absolute value. Let

$$f_n(Z_1, Z_2) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{p\left(Z_1^k, Z_2^k\right)}{Z_1^{k\,j_1} Z_2^{k\,j_2}}.$$

Then $f_n\left(a_1^k, a_2^k\right) = 0$ for all integers k. Consider how each monomial in p contributes to f_n. A monomial $Z_1^{\ell_1} Z_2^{\ell_2}$ yields a term

$$\frac{1}{n} \sum_{k=0}^{n-1} Z_1^{k\,(\ell_1 - j_1)} Z_2^{k(\ell_2 - j_2)}.$$

Write $a_j = r_j e^{i\theta_j}$. At $(Z_1, Z_2) = (a_1, a_2)$, this term yields

$$\frac{1}{n} \sum_{k=0}^{n-1} r_1^{k\,(\ell_1 - j_1)} r_2^{k(\ell_2 - j_2)} e^{ik(\ell_1 - j_1 + \ell_2 - j_2)}.$$

This term goes to 0 as $n \to \infty$ unless $r_1^{\ell_1 - j_1} r_2^{\ell_2 - j_2} = 1$ and $\ell_1 - j_1 + \ell_2 - j_2$ is a multiple of 2π, i.e. vanishes. In particular, the term coming from the monomial $Z_1^{j_1} Z_2^{j_2}$ yields a nonzero contribution in the limit. But in the limit $f_n(a_1, a_2) \to 0$, so some other monomial must cancel $Z_1^{j_1} Z_2^{j_2}$. Therefore there must be some pairs (j_1, j_2) and (ℓ_1, ℓ_2) for which $a_1^{j_1} a_2^{j_2} = a_1^{\ell_1} a_2^{\ell_2}$. So g is hyperresonant. The terms in f_n which don't vanish in the limit as $n \to \infty$ must all have powers of Z_1 and Z_2 differing from (j_1, j_2) by integer multiples of the hyperresonance of g.

We can grade each monomial $Z_1^{k_1} Z_2^{k_2}$, by sliding (k_1, k_2) over by integer multiples of the hyperresonance until we make k_1 as small as possible, and using the resulting k_1 value as the grading. We can write each polynomial $p(Z_1, Z_2)$ as a sum of graded pieces. Suppose $p(Z_1, Z_2)$ vanishes on all of the

points $\left(a_1^k, a_2^k\right)$. Let's write $p\left(Z_1, Z_2\right) = \sum p_j\left(Z_1, Z_2\right)$ as a sum of graded pieces. Consider again these functions $f_n\left(Z_1, Z_2\right)$. Taking the limit

$$0 = \lim_{n \to \infty} f_n\left(a_1, a_2\right)$$

only the terms from the highest graded piece enter into the limit. If

$$p_N\left(Z_1, Z_2\right) = \sum_{\ell_1, \ell_2} c_{\ell_1 \ell_2} Z_1^{\ell_1} Z_2^{\ell_2}$$

is the highest graded piece, then

$$0 = \lim_{n \to \infty} f_n\left(a_1, a_2\right) = \sum c_{\ell_1 \ell_2}.$$

Modulo the hyperresonance relation $Z_1^{n_1} - Z_2^{n_2}$, each term in $p_N\left(Z_1, Z_2\right)$ can be shifted over to become a multiple of one single term:

$$p_N\left(Z_1, Z_2\right) = Z_1^{N_1} Z_2^{N_2} \sum_{\ell_1, \ell_2} c_{\ell_1 \ell 2} = 0.$$

$\square$

Let $H_0 \subset \mathrm{GL}\left(2, \mathbb{C}\right)$ be the subgroup fixing a point of $\mathbb{C}^2 \setminus 0$. There is an obvious Lie group morphism $g \in \mathrm{GL}\left(2, \mathbb{C}\right) \mapsto (g, 0) \in G$ where as above

$$G = \left(\mathrm{GL}\left(2, \mathbb{C}\right) / n\text{-th roots of } 1\right) \rtimes \mathrm{Sym}^n \left(\mathbb{C}^2\right)^*.$$

Lemma 8.2. *Every $\mathcal{O}(n)$-structure on any linear Hopf surface is induced by its $\mathrm{GL}\left(2, \mathbb{C}\right)/H_0$-structure.*

Proof. We map $\Phi : \mathbb{C}^2 \setminus 0 \to \mathcal{O}(n)$ by the identity map in affine coordinates, and then map $\Phi : \mathrm{GL}\left(2, \mathbb{C}\right) \to G$ by the embedding $g \mapsto (g, 0)$. $\square$

Lemma 8.3. *The holonomy group of any $\mathcal{O}(n)$-structure on any Hopf surface S_F is contained in $\mathrm{SL}\left(2, \mathbb{C}\right)/n$-th roots of 1, precisely if the $\mathcal{O}(n)$-structures is the eigenstructure of*

1. *a hyperresonant linear map F with eigenvalues λ_1 and λ_2 and hyperresonance either*

   (a) $\lambda_1^n = \lambda_2^2$ *or*

   (b) $\lambda_1^{n/2} = \lambda_2$,

   or

2. *a nondiagonalizable linear map F when $n = 2$, i.e. an $\mathcal{O}(2)$-structure.*

Proof. Just take determinants of the holonomy generators. Note that there are two distinct eigenstructures for a diagonalizable linear map F, corresponding to the two distinct eigenspaces. $\square$

Proposition 8.4. *Suppose that some G'/H'-structure induces the* $\mathrm{GL}(2,\mathbb{C})/H_0$*-structure on some linear Hopf surface. Then* $G' \to \mathrm{GL}(2,\mathbb{C})$ *is onto or else* $G' \to \mathrm{SL}(2,\mathbb{C})$ *is onto.*

Proof. There are no other subgroups of $\mathrm{GL}(2,\mathbb{C})$ which act transitively on $\mathbb{C}^2\backslash 0$; see Huckleberry and Livorno [14]. □

Remark 8.5. The eigenstructure on a linear Hopf surface S_F is induced, as we have already proven, by the G_0/H_0-structure, where G_0 is the group of linear transformations of $\mathbb{C}^2$ preserving an eigenspace of the linear map F.

Lemma 8.6. *The radial structure on the generic Hopf surface is induced by the* $\mathrm{GL}(2,\mathbb{C})/H_0$*-structure on* $\mathbb{C}^2\backslash 0$ *and by no proper subgroup of* $\mathrm{GL}(2,\mathbb{C})$.

Proof. For a generic Hopf surface, the radial and eigen structures will have holonomy generator Zariski dense in the diagonal matrices. That ensures that for any holomorphic reduction, say to a G'/H'-structure, G' will have to map onto a subgroup of G containing the diagonal matrices. Moreover, G' will have an open orbit in $\mathcal{O}(n)$, containing at least the open orbit of the diagonal matrices, which is everything except the fibers over 0 and ∞ and the 0-section. However, the radial structure has everything but the 0-section in its image, so we will need G' to have as image a larger group than just the diagonal subgroup. Indeed our group G' will need to act transitively on $\mathbb{P}^1$, so must map onto $\mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ by the classification of homogeneous surfaces (see Huckleberry and Livorno [14]). Any subgroup of $\mathrm{GL}(2,\mathbb{C})/n$-th roots of 1 mapping onto $\mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ will have to contain $\mathrm{SL}(2,\mathbb{C})/n$-th roots of 1 and therefore the image of G' must contain all of $\mathrm{GL}(2,\mathbb{C})/n$-th roots of 1. □

9 Conclusions

We have found all of the $\mathcal{O}(n)$-structures on all Hopf surfaces explicitly, by computed their developing maps and holonomy groups explicitly. This makes it possible to determine which of these structures are induced from other locally homogeneous geometric structures on Hopf surfaces. The one surprising result of the classification is the appearance of the hyperresonant $\mathcal{O}(n)$-structures (on the hyperresonant Hopf surfaces). The hyperresonant $\mathcal{O}(n)$-structures have no apparent geometric or intuitive description. They depend on the prescence of complicated meromorphic functions (rational functions in the canonical affine structure), and so disappear on the Hopf surfaces with trivial function fields.

The relation of these results to Wall's results [31, 32] deserves some clarification. The full picture, of all holomorphic locally homogeneous geometric structures on compact complex surfaces, and which are induced from which, is still hidden. It seems likely that this picture will soon become clear. The classification of holomorphic Cartan geometries on compact complex surfaces would then appear to be within reach. We have to keep in mind that the explicit classification of holonomy morphisms and developing maps for holomorphic projective connections on complex algebraic curves is still unknown, and

perhaps too complicated to be classifiable (see [8]). Therefore we would only hope to classify holomorphic Cartan geometries on compact complex surfaces modulo the classification on curves. It seems likely that holomorphic Cartan geometries can be classified on linear Hopf manifolds in all dimensions.

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